The twin paradox revisited

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The twin paradox of special relativity results from failing to recognize the fundamental asymmetry between the two twins: One and only one twin changes direction and thus undergoes acceleration. What happens if the universe is closed, so that the twin in the rocket can return to Earth *without* changing direction? The twin paradox is reformulated in a closed universe (a cylinder) and then resolved by showing that there is still a (much more subtle!) asymmetry between the two twins.

I. INTRODUCTION

The twin paradox is perhaps the best-known paradox of special relativity. At the heart of the twin paradox is the asymmetry between the "stay-at-home" twin and the "rocket" twin caused by the fact that the "stay-at-home" twin has *constant* velocity, while the "rocket" twin must change direction, thus undergoing (in the usual formulation) infinite acceleration at one instant of time. This asymmetry can be removed by considering a closed universe in which it is possible to return to the starting point by traveling in a "straight" line. Now, it seems, we truly have a paradox! Which twin is older when they meet again?

We first briefly review the usual twin paradox in Sec. II, then modify it in Sec. III for a closed, two-dimensional universe (a cylinder). After stating the modified version of the twin paradox, we then show in Sec. IV via a direct algebraic calculation (which may be skipped on first reading) that the "rocket" twin is indeed the younger, and that there is no paradox. In Sec. V we give a purely geometric resolution of the paradox that emphasizes the underlying concepts. Finally, for those readers who have some familiarity with general relativity, we present some additional aspects of our cylindrical universe in Sec. VI.

II. THE TWIN PARADOX

According to special relativity, an observer Z' moving in the x direction with constant velocity v measures length x' and time t' as given by the Lorentz transformation,

$$t' = t\cosh\beta - x\sinh\beta; \quad x' = -t\sinh\beta + x\cosh\beta, \tag{1}$$

where $v = \tanh\beta$ and where we work in units such that the speed of light c equals 1. This leads to the well-known effect of time dilation: An observer Z at rest in the (x,t) coordinate system sees a clock carried by Z' run more slowly than his own: Elapsed time from the origin $x = 0 = t$ along $x = 0$ is given by $\Delta t' = \Delta t \cosh\beta > \Delta t$.

However, there is nothing preferred about the observer Z as opposed to Z'. From the point of view of Z', it is Z' who is at rest and Z who is moving with constant velocity $-v$. This is expressed mathematically by noting that (1) can easily be inverted to give

$$t = t'\cosh\beta + x'\sinh\beta; \quad x = t'\sinh\beta + x'\cosh\beta. \tag{2}$$

Thus Z' sees a clock carried by Z run more slowly than her own: $\Delta t = \Delta t' \cosh\beta > \Delta t'$ along $x' = 0$.

There is no paradox here, as can perhaps be emphasized by the following analogy:[1] One mile "due north" with respect to the North Pole corresponds to less than 1 mile "due north" with respect to magnetic north (because there will be a slight east/west component). But conversely, 1 mile "due north" with respect to magnetic north corresponds to less than 1 mile "due north" with respect to the North Pole. The only paradox here is the unfortunate use of the word "north" to refer to two different directions; the only paradox in time dilation is the unfortunate Newtonian preconception that there is a preferred time direction.

The twin paradox comes about from trying to compare the clocks carried by two different observers, and thus their ages. A typical version goes as follows.[2,3] Suppose that twin A, the "stay-at-home" twin, does just that and remains at home on Earth. Meanwhile, his twin sister A′, the "rocket" twin, goes on a fast trip in a powerful rocket to the nearest star and back. See Fig. 1. Because of time dilation, A sees A′ as aging more slowly, and thus concludes that he will be older than his sister A′ when she returns. But wait a minute! A′ sees the Earth move away and return, and hence sees A as aging more slowly! She, therefore, predicts that her brother A will be younger!

All of the above statements are correct except the final conclusion. The resolution of the twin paradox is that only one of the twins accelerates, namely, the "rocket" twin A′. In the usual formulation, this acceleration takes place at the instant of time when A′ suddenly reverses her direction of travel upon reaching the star. At that "instant" of time, A′ changes from one inertial coordinate system to another (while undergoing infinite acceleration). A correct description of the situation as seen by A′ must take into account the fact that only A is truly an inertial (nonaccelerating) observer. When this is done,[2] both A and A′ will predict that A′ will be younger.

III. THE TWIN PARADOX ON A CYLINDER[3]

Consider now a two-dimensional universe in the shape of an infinitely long cylinder with circumference λ, so that time t runs "up" the cylinder, and space x runs "around" the cylinder. Thus, at any instant of time ($t = \text{const}$), the universe looks like a circle with circumference λ. We have changed the topology of space-time from that of a piece of paper ($\mathbb{R}^2$) to that of a cylinder ($S^1 \times \mathbb{R}^1$). However, this is a *global* change that does not affect the local structure of space-time. In particular, a cylinder is flat precisely because it can be made from a piece of paper without stretching or tearing.[4] Thus special relativity continues to describe the physics in such a universe.[5]

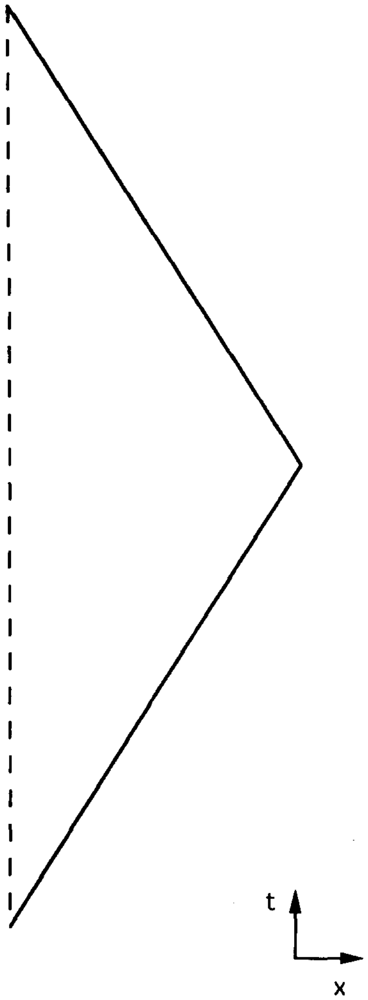

Fig. 1. The space-time paths of the "rocket" twin (solid line) and the "stay-at-home" twin (dashed line) in the usual twin paradox.

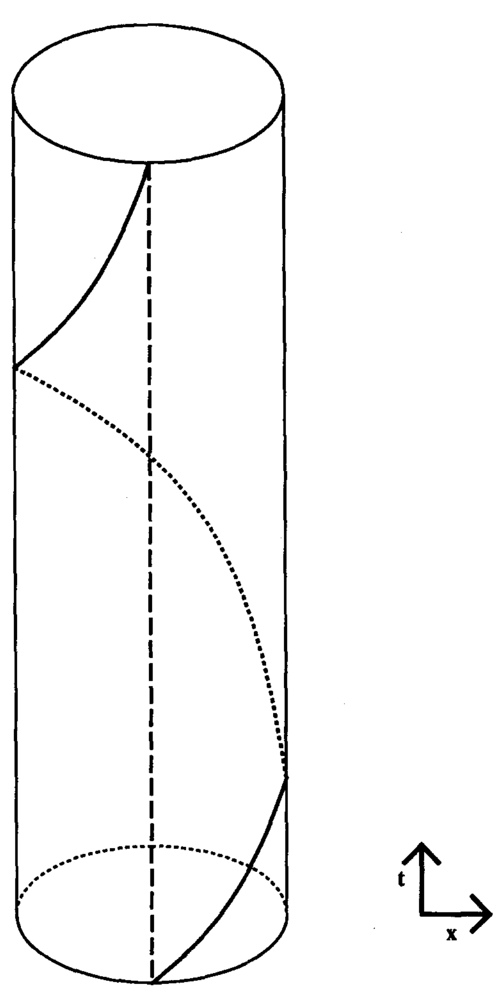

Fig. 2. The space-time paths of the "rocket" twin (solid/dotted line) and the "stay-at-home" twin (dashed line) in a cylindrical universe.

Now consider the twins B and B′, one of whom, B, remains at rest in space ($x = \text{const}$), and thus moves straight up the cylinder in space-time, while the other, B′, moves at constant velocity and thus moves around the cylinder. See Fig. 2. Since B sees B′ as aging more slowly, the "stay-at-home" twin B again sees the "rocket" twin B′ as being the younger when the latter returns after having circumnavigated the universe. But, from the point of view of the "rocket" twin B′, it is the "stay-at-home" twin B who has circumnavigated the universe while B′ has remained at rest! Surely B′ is also entitled to conclude that B is younger, and we seem to have a real paradox.

IV. ALGEBRAIC RESOLUTION

The cylinder can be thought of as being formed from the rectangular region in the plane given by $x \in [0,\lambda]$ with the lines $x = 0$ and $x = \lambda$ being identified. We can again define x', t' by (1) so long as we restrict ourselves to this domain for x. As a result, we have $x' \in [-vt', -vt' + \lambda/\gamma]$ where $\gamma = 1/\sqrt{1-v^2} = \cosh\beta$. Figure 3(a) shows the cylinder as seen by the "stay-at-home" twin B, while Fig. 3(b) shows the cylinder as seen by the "rocket" twin B′. O denotes the origin $x = t = 0 = x' = t'$ and is to be identified with R; $\overline{\text{OR}}$ is a circle of constant time t. P denotes the point at which B′ comes back to Earth. $\overline{\text{OP}}$ shows the trajectory of the "rocket" twin B′, while $\overline{\text{OQ}}$ is (part of) a helix of (locally) constant time t'.[6]

The elapsed proper time along any path as measured by a clock carried along the path is just the (Lorentzian) length of the path. Thus the elapsed time T as seen by the "stay-at-home" twin B is the length of the line segment $\overline{\text{RP}}$, which is $T = \lambda/v \equiv \lambda/\tanh\beta$. On the other hand, the elapsed time T' as seen by the "rocket" twin B′ is the length of $\overline{\text{OP}}$, which is $T' = \lambda/v\gamma \equiv \lambda/\sinh\beta$ so that

$$T = \gamma T' \equiv T'\cosh\beta > T'. \tag{3}$$

We have thus established that the "stay-at-home" twin B does indeed age more than the "rocket" twin B′.

How do the twins themselves see things? The "stay-at-home" twin B sees his sister B′ moving with constant velocity v along the worldline $\overline{\text{OP}}$. Therefore, he concludes that his sister's clock runs slower, so that B′ should be younger than B when she returns. So far, so good. On the other hand, the "rocket" twin B′ sees her brother B moving with constant velocity $-v$ from O to P, and thus must also see her brother's clock run slower, so that B ought to be the younger.

The key point is that B′ sees B move from O to P *along the worldline* $\overline{\text{RP}}$. This means that B′ sees B move a spatial distance $\Delta x' = \lambda\gamma$ in time $\Delta t' = \lambda/v\gamma + \lambda v\gamma = \lambda\gamma/v$, and

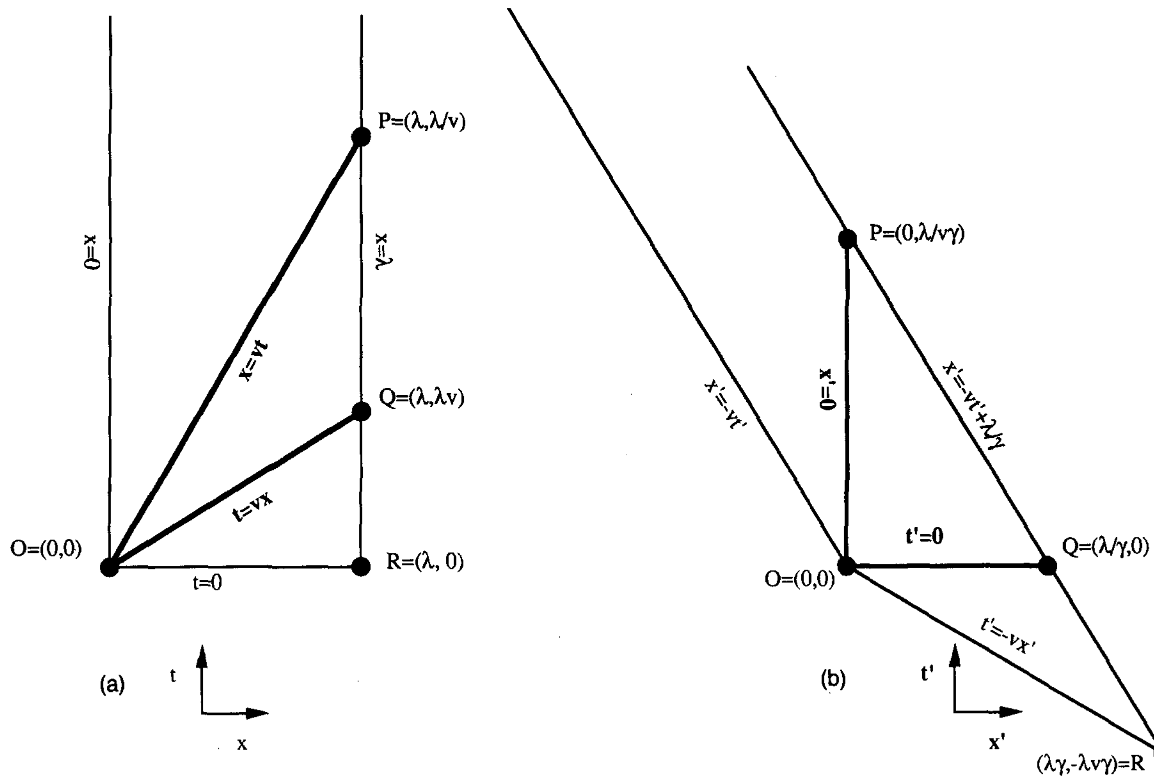

Fig. 3. The cylindrical universe as seen by (a) the "stay-at-home" twin B and (b) the "rocket twin" B'. O and R represent the same point, namely, where the twins separate, and P is where they rejoin each other. The coordinates (x,t) of B are related to the coordinates (x',t') of B' by Eqs. (1) and (2) with $v = \tanh \beta$.

not a distance λ/γ (which is the apparent circumference $\overline{OQ}$ of the cylinder for B'). Therefore, B' thinks B's clock reads $(\Delta x')^2 - (\Delta t')^2 = \lambda/v$, which it does $(t|_P = \lambda/v)$. Similarly, B sees B' move a distance $\Delta x = \lambda$ in time $\Delta t = \lambda/v$ and therefore thinks B''s clock reads $(\Delta x)^2 - (\Delta t)^2 = \lambda/v\gamma$, which it does $(t'|_P = \lambda/v\gamma)$. Both twins therefore agree on the calculated age of each twin.

V. GEOMETRIC RESOLUTION

One of the lessons of relativity is that one should attempt to think geometrically whenever possible. This forces one to formulate the paradox in terms of invariant concepts as opposed to observer-dependent concepts.

In the case at hand, the necessary invariant concept is that of proper time: the time measured along a path in space-time by an observer traveling along the path. This is just the "length" of the path, measured with respect to the Lorentzian concept of interval as opposed to the Euclidean concept of distance. The Lorentzian signature implies that the triangle inequality goes the wrong way: Whereas in Euclidean geometry one side of a triangle is always shorter than the sum of the other two, in Lorentzian geometry the direct (timelike) path between two points always takes *more* time than any indirect (timelike) path.

From this point of view, the usual twin paradox is easily resolved: One simply measures the "length" of the paths of the two twins A and A'. In the usual formulation, A moves in a straight line, whereas A' moves along two legs of a triangle. Thus A moves along the "shorter" path and therefore takes more time, so that A' is younger when they rejoin each other. Note that this method easily deals with an arbitrary path for A': The "shortest" path is clearly that taken by A, so A is always older.

For our modified twin paradox things are just as simple: A line parallel to the sides of a cylinder is clearly shorter than one that spirals around it. Thus twin B moves along the "shorter" path, and hence is older.

VI. DISCUSSION

For those with some knowledge of general relativity we point out some additional properties of our cylindrical universe. *Both* observers B and B' see the universe as being not only stationary but also static, despite the fact that the surfaces that locally have the form $\{t' = \text{const}\}$ are helices and not circles. This is different from the case in three or more dimensions where rotating observers are stationary but not static. This illustrates a *global* breakdown in the concept of "surfaces of constant time."[7]

This global problem means that, even though the cylinder is Lorentz invariant (i.e., Lorentz transformations preserve the notion of space-time interval $[(\Delta x)^2 - (\Delta t)^2]$, there is nevertheless a preferred family of observers, namely, those parallel to B. In fact, this family can be *defined* by the property that its observers are always the oldest in any twin paradox calculation between different stationary observers.[8] Alternatively, such observers may be singled out by noting that they are the only stationary observers who cannot distinguish between "forward" and "backward"; e.g., by sending light rays in both directions around the cylinder and seeing which returns first. Thus, unlike in ordinary spatial relativity, there *is* a preferred time direction in the cylindrical universe![9]

A related effect occurs if a clock is carried around the globe; again, a moving clock measures less time than one at rest. However, since the Earth itself is rotating, a clock on the surface of the Earth is *not* at rest. One may nevertheless conclude that a clock carried around the world with the Earth's rotation (to the east) will lose time compared to a clock at rest on the surface of the Earth whereas a clock carried against the rotation (to the west) will gain time. This experiment was actually done in 1971 using atomic clocks on commercial flights.[10] There are two effects here. The one described above is due solely to special relativity and is called the Sagnac effect. There is also a general relativistic "gravitational redshift" due to the difference in altitude between the clocks (and thus a difference in the strength of gravitational field seen by the clocks); note that this latter effect is the same in both directions. A similar experiment was done later using electromagnetic signals instead of clocks.[11]

Other authors have considered twins in different, freely falling trajectories in the gravitational field of the Earth,[12] and corevolving versus counterrevolving twins near a Kerr (rotating) black hole.[13]

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[1]Edwin F. Taylor and John Archibald Wheeler, *Spacetime Physics* (Freeman, San Francisco, 1963, 1966).

[2]For a very nice and much more detailed treatment of the twin paradox, see Bernard F. Schutz, *A First Course in General Relativity* (Cambridge U.P., Cambridge, 1985). See also Robert Perrin, "Twin paradox: A complete treatment from the point of view of each twin," Am. J. Phys. **44**, 317–319 (1970) and W. G. Unruh, "Parallax distance, time, and the twin 'paradox,' " Am J. Phys. **49**, 589–592 (1981).

[3]This problem has been previously considered by Carl H. Brans and Dennis Ronald Stewart, "Unaccelerated-Returning-Twin Paradox in Flat Space-Time," Phys. Rev. D **8**, 1662–1666 (1973) and, more recently, by R. J. Low, "An acceleration-free version of the clock paradox," Eur. J. Phys. **11**, 25–27 (1990). Although our presentations are somewhat different, the main results are the same.

[4]A sphere is *not* flat because it cannot be covered by a piece of paper. This is why map projections are so complicated!

[5]General relativity describes the physics of curved space-times and reduces to special relativity if the space-time is not curved.

[6]Remember that Q is *not* the same as O!

[7]Roughly speaking, a stationary observer is one for whom the universe remains constant in time, whereas a static observer must also see well-defined *local* surfaces of constant time.

[8]Steven G. Harris (private communication).

[9]P. C. Peters, "Periodic boundary conditions in special relativity," Am. J. Phys. **51**, 791–795 (1983).

[10]J. C. Hafele and R. E. Keating, "Around-the-world atomic clocks: Predicted relativistic time gains," Science **177**, 166–168 (1972). J. C. Hafele and R. E. Keating, "Around-the-world atomic clocks: Observed relativistic time gains," Science **177**, 168–170 (1972).

[11]D. W. Allan, M. A. Weiss, and N. Ashby, "Around-the-world relativistic Sagnac effect," Science **228**, 69–70 (1985).

[12]Barry R. Holstein and Arthur R. Swift, "The relativity twins in free fall," Am. J. Phys. **40**, 746–750 (1972).

[13]F. Landis Markley, "Relativity twins in the Kerr metric," Am. J. Phys. **41**, 1246–1250 (1973); Donald E. Hall, "Can local measurements resolve the twin paradox in a Kerr metric," Am. J. Phys. **44**, 1204–1208 (1976).

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Orbits are not difficult to comprehend. It is gravity which stirs the depths of insomnia.

Of a Fire on the Moon (Little, Brown, Boston, 1970).