MTH 437/537 **HW #7** Spring 2024

0. **WARMUP** *(Do not turn this problem in, but see me if you need help.)*

Determine the (nonzero) components ${R^i}_{jkl}$ of the curvature 2-forms

$$\Omega^i j = \frac{1}{2}{R^i}_{jkl}\,\sigma^k \wedge \sigma^l$$

for the Robertson–Walker geometry, with line element

$$ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2\right)$$

with $k = -1, 0, 1$ depending on whether the spatial cross-sections are hyperbolic, flat, or spherical, respectively.

One solution can be found in §A.9 of DFGGR – but try it on your own first!

1. **TRACES**

The components of the Einstein vector-valued 1-form $\vec{\boldsymbol{G}} = {G^i}_j\sigma^j\hat{\boldsymbol{e}}_i$ are related to those of the Ricci vector-valued 1-form $\vec{\boldsymbol{R}} = {R^i}_j\sigma^j\hat{\boldsymbol{e}}_i$ by

$${G^i}_j = {R^i}_j - \frac{1}{2}\,{\delta^i}_j\,R,$$

where $R = {R^i}_i$. Find an expression for the trace $G = {G^i}_i$ of $\vec{\boldsymbol{G}}$ in terms of R.

*R is called the **trace** of $\vec{\boldsymbol{R}}$; more precisely, it is the trace of the matrix of components $\left({R^i}_j\right)$.*

You may assume if desired that the underlying geometry is 4-dimensional, with signature 1.

2. **ROBERTSON–WALKER GEOMETRY**

The components of the Ricci vector-valued 1-form $\vec{\boldsymbol{R}}$ (given above) are related to the components of the curvature 2-form ${\Omega^i}_j$ (also given above) by

$$R_{ij} = {R^m}_{imj}$$

where the lowering of indices is accomplished with the metric, that is, $R_{ij}\sigma^j = \hat{\boldsymbol{e}}_i \cdot \vec{\boldsymbol{R}}$. Compute the (nonzero) components ${G^i}_j$ of the Einstein vector-valued 1-form for the Robertson–Walker geometry.

You do not need to complete the Warmup question above before attempting this problem, although you may wish to do so.