MTH 437/537 **HW #7** Spring 2021

0. **WARMUP** *(Do not turn this problem in, but see me if you need help.)*
Determine the (nonzero) components $R^i{}_{jkl}$ of the curvature 2-forms

$$\Omega^i j = \frac{1}{2} R^i{}_{jkl}\, \sigma^k \wedge \sigma^l$$

for the Robertson–Walker geometry, with line element

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \right)$$

with $k = -1, 0, 1$ depending on whether the spatial cross-sections are hyperbolic, flat, or spherical, respectively.
One solution can be found in §A.9 of DFGGR – but try it on your own first!

1. **TRACES**
The components of the Einstein vector-valued 1-form $\vec{\boldsymbol{G}} = G^i{}_j \sigma^j \hat{\boldsymbol{e}}_i$ are related to those of the Ricci vector-valued 1-form $\vec{\boldsymbol{R}} = R^i{}_j \sigma^j \hat{\boldsymbol{e}}_i$ by

$$G^i{}_j = R^i{}_j - \frac{1}{2}\, \delta^i{}_j\, R,$$

where $R = R^i{}_i$. Find an expression for the trace $G = G^i{}_i$ of $\vec{\boldsymbol{G}}$ in terms of R.
*R is called the **trace** of $\vec{\boldsymbol{R}}$; more precisely, it is the trace of the matrix of components $\left(R^i{}_j\right)$.*
You may assume if desired that the underlying geometry is 4-dimensional, with signature 1.

2. **ROBERTSON–WALKER GEOMETRY**
The components of the Ricci vector-valued 1-form $\vec{\boldsymbol{R}}$ (given above) are related to the components of the curvature 2-form $\Omega^i{}_j$ (also given above) by

$$R_{ij} = R^m{}_{imj}$$

where the lowering of indices is accomplished with the metric, that is, $R_{ij}\sigma^j = \hat{\boldsymbol{e}}_i \cdot \vec{\boldsymbol{R}}$. Compute the (nonzero) components $G^i{}_j$ of the Einstein vector-valued 1-form for the Robertson–Walker geometry.
You do not need to complete the Warmup question above before attempting this problem, although you may wish to do so.