MTH 437/537 **HW #6** Spring 2021

1. **SHAPE OF SCHWARZSCHILD GEOMETRY**

(a) Plot the graph of $8mr = h^2 + 16m^2$ with r as the *horizontal* coordinate.
Equivalently, solve this equation for h, and plot h as a function of r.

(b) Express arclength along this graph in terms of r (and dr).
Assume that (r, h) are rectangular, Euclidean coordinates.

(c) Consider the surface of revolution obtained by rotating your curve about the h-axis. What is the line element for this surface? *Compare your answer with the Schwarzschild geometry.*

(d) Find the Gaussian curvature of this surface. Is it positive or negative?

2. **INDEX GYMNASTICS**

In a coordinate basis $\{dx^i\}$ of 1-forms, the components g_{ij} of the metric are defined by $ds^2 = g_{ij}\, dx^i\, dx^j$. The dual basis $\{\vec{\boldsymbol{e}}_i\}$ of vectors satisfies $\vec{\boldsymbol{e}}_i \cdot \vec{\boldsymbol{e}}_j = g_{ij}$.

*These bases are **not** necessarily orthogonal. It is however still true that $d\vec{\boldsymbol{r}} = dx^i\, \vec{\boldsymbol{e}}_i$.*

(a) Determine an expression for $\vec{\boldsymbol{e}}_i \cdot \vec{\nabla} f$ in terms of partial derivatives.

(b) Acting on 1-forms $F = \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$, $G = \vec{\boldsymbol{G}} \cdot d\vec{\boldsymbol{r}}$, the metric satisfies $g(F, G) = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{G}}$ for any vectors $\vec{\boldsymbol{F}}$, $\vec{\boldsymbol{G}}$. Express the components $g^{ij} = g(dx^i, dx^j)$ in terms of the components g_{ij}.
A derivation in 2 dimensions is acceptable for MTH 437 students. For the general case, use linear algebra rather than attempting to provide an explicit formula for each component.

LOOKING AHEAD

The following warmup problem will appear on next week's assignment. It will not be graded (this week or next), but you may wish to get a head start on it.

Determine the curvature 2-forms for the Robertson–Walker geometry, with line element

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \right)$$

with k constant.

One solution can be found in §A.9 of DFGGR – but try it on your own first!