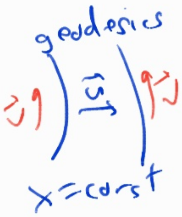
Geodesic Deviation

Idea: connect curvature to geodesic separation



$$\dot{\vec v}=0,\quad |\vec v|^2=-1 \quad \therefore\ \vec v=\hat\tau$$

$$d\vec r = \underbrace{d\tau}_{\sigma^\tau}\,\hat\tau + \underbrace{\sigma^x}_{h\,dx}\,\hat x + \dots$$

$$\Rightarrow \vec u = h\hat x \Rightarrow \vec u\cdot\vec\nabla f = \frac{\partial f}{\partial x}$$

$$\Rightarrow d\vec r = \vec v\,d\tau + \vec u\,dx$$

Lemma: $\dot{\vec u} = \vec v\,'$

Pf: $0 = d^2\vec r = d\vec v\wedge d\tau + d\vec u\wedge dx$

torsion free

$$= \vec v\,' dx\wedge d\tau + \dot{\vec u}\, d\tau\wedge dx \quad ✓$$

$$\Rightarrow d\vec v = \vec v\,' dx + \cancel{\dot{\vec v}}^{\,0} d\tau$$

$$= \dot{\vec u}\,dx$$

$$d^2\vec v = \ddot{\vec u}\, d\tau\wedge dx$$

$$d^2\vec v = d^2\hat\tau = \Omega^i{}_\tau\, \hat e_i = \Omega^x{}_\tau\, \hat x$$

$$= \tfrac12 R^x{}_{\tau i j}\,\sigma^i\wedge\sigma^j\, \hat x$$

$$= R^x{}_{\tau x\tau}\,\underbrace{\sigma^x}_{h\,dx}\wedge\sigma^\tau\, \hat x$$

$$= R^x{}_{\tau x \tau}\, dx\wedge d\tau\, \vec u$$

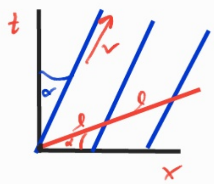
$$\Rightarrow \boxed{\ddot{\vec u} = -R^x{}_{\tau x\tau}\,\vec u}$$

Dust

SWBQ: Draw 3 (identical, equally spaced) objects moving to the right (at the same speed).

① What is their velocity?

② How far apart are they?



speed : $\tanh\alpha$

vel : $|\vec{v}|^2 = -1$

$\vec{v} = \sinh\alpha\,\hat{x} + \cosh\alpha\,\hat{t}$

separation : ℓ in rest frame

$\bar{\ell} = \ell/\cosh\alpha$ in lab frame

③ How much energy does each object have?

mom: $m\vec{v} = mc\sinh\alpha\,\hat{x} + mc^2\cosh\alpha\,\hat{t}$

"3mom" "energy"

rest: $E = mc^2$

lab: $\bar{E} = mc^2\cosh\alpha$

④ What is the energy density?

$\rho = E/\ell <$ rest: mc^2/ℓ ; lab: $mc^2\cosh^2\alpha/\ell$

$\rho_{rest} = E/\ell$

$\rho_{lab} = \bar{E}/\bar{\ell} = \dfrac{E\cosh\alpha}{\ell/\cosh\alpha}$

Stress - Energy - Momentum

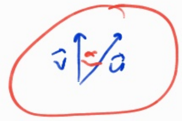
$$\boxed{\vec{T} = T^{i}{}_{\partial}\, \sigma^{\partial}\, \hat{e}_{i}}$$

vector-valued 1-form

Dust: $u = \vec{u}\cdot d\vec{r} = -d\lambda$ ← dust proper time

$$\boxed{\vec{T} = \rho\, u\, \hat{u}}$$

Observer: $v = \vec{v}\cdot d\vec{r} = -d\tau$ ← observer proper time



Energy density

at rest $\quad g(u,\vec{T})\cdot\vec{u} = \overbrace{\rho\, \underbrace{g(u,u)}_{-1}\, \underbrace{\vec{u}\cdot\vec{u}}_{-1}}^{=\rho}$

lab $\quad g(\vec{T},v)\cdot\vec{v} = \rho\, \underbrace{g(u,v)}_{-\cosh\alpha}\, \underbrace{\vec{u}\cdot\vec{v}}_{-\cosh\alpha}$

$$= \rho\cosh^2\alpha$$