HW #6

1. **INDEX GYMNASTICS**

In a coordinate basis $\{dx^i\}$ of 1-forms, the components g_{ij} of the metric are defined by $ds^2 = g_{ij}\,dx^i\,dx^j$. The dual basis $\{\vec{\boldsymbol{e}}_i\}$ of vectors satisfies $\vec{\boldsymbol{e}}_i \cdot \vec{\boldsymbol{e}}_j = g_{ij}$.

These bases are ***not*** *necessarily orthogonal. It is however still true that* $d\vec{\boldsymbol{r}} = dx^i\,\vec{\boldsymbol{e}}_i$.

(a) Determine an expression for $\vec{\boldsymbol{e}}_i \cdot \vec{\nabla} f$ in terms of partial derivatives.

(b) Acting on 1-forms $F = \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$, $G = \vec{\boldsymbol{G}} \cdot d\vec{\boldsymbol{r}}$, the metric satisfies $g(F, G) = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{G}}$ for any vectors $\vec{\boldsymbol{F}}$, $\vec{\boldsymbol{G}}$. Express the components $g^{ij} = g(dx^i, dx^j)$ in terms of the components g_{ij}.

A derivation in 2 dimensions is acceptable if you don't see how to handle the general case.

2. **DOUBLE-NULL COORDINATES**

In 2-dimensional Minkowski space, let $u = t - x$, $v = t + x$.

(a) Express the line element $ds^2 = -dt^2 + dx^2$ in terms of the coordinate basis $\{du, dv\}$.

(b) Determine the components g^{ij} in this basis.

(c) Compute $g_{ij}g^{jk}$. What sort of a beast is it the components of?

3. **TRACES**

Suppose that two vector-valued 1-forms $\vec{\boldsymbol{G}} = G^i{}_j \sigma^j \vec{\boldsymbol{e}}_i$ and $\vec{\boldsymbol{R}} = R^i{}_j \sigma^j \vec{\boldsymbol{e}}_i$ have components that are related by

$$G^i{}_j = R^i{}_j - \frac{1}{2}\,\delta^i{}_j\,R,$$

where $R = R^i{}_i$. Find an expression for the trace $G = G^i{}_i$ of $\vec{\boldsymbol{G}}$ in terms of R.

R is called the ***trace*** *of $\vec{\boldsymbol{R}}$; more precisely, it is the trace of the matrix of components $\left(R^i{}_j\right)$.*

You may assume if desired that the underlying geometry is 4-dimensional, with signature 1.