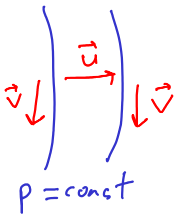
Geodesic Deviation



Geodesic

$$\dot{\vec v} = 0;\ |\vec v|^2 = -1 \Rightarrow \vec v = \hat\tau$$

$$\Rightarrow ds^2 = -d\tau^2 \Rightarrow d\vec r = d\tau\,\hat\tau$$

Family of geodesics

$$dp \perp \{p = \text{const}\}$$

$$\Rightarrow ds^2 = -d\tau^2 + h^2\,dp^2$$

$$\text{Set } \vec u = h\,\hat p \Leftrightarrow \vec u\cdot\vec\nabla f = \frac{\partial f}{\partial p}$$

$$\Rightarrow d\vec r = d\tau\,\hat\tau + h\,dp\,\hat p$$

$$= \vec v\,d\tau + \vec u\,dp$$

Lemma: $\boxed{\dot{\vec u} = \vec v\,'}$

Pf: $\underbrace{0}_{\text{torsion free}} = d^2\vec r = d\vec v\wedge d\tau + d\vec u\wedge dp$

$$= \vec v\,'\,dp\wedge d\tau + \dot{\vec u}\,d\tau\wedge dp \quad ✓$$

$$\therefore\ d\vec v = \cancelto{0}{\dot{\vec v}}\,d\tau + \vec v\,'\,dp = \dot{\vec u}\,dp$$

$$\Rightarrow d^2\vec v = \ddot{\vec u}\,d\tau\wedge dp$$

But

$$d^2\vec v = d^2\hat\tau = \Omega^i{}_\tau\,\hat e_i = \Omega^p{}_\tau\,\hat p$$

$$= \tfrac12 R^p{}_{\tau i j}\,\sigma^i\wedge\sigma^j\,\hat p$$

$$= R^p{}_{\tau p \tau}\,h\,dp\wedge d\tau\,\hat p \qquad \text{(no sum)}$$

$$= R^p{}_{\tau p\tau}\,dp\wedge d\tau\,\vec u$$

$$\Rightarrow \boxed{\ddot{\vec u} = -R^p{}_{\tau p \tau}\,\vec u}$$