HW #4

All references to black holes assume the Schwarzschild line element:

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

All orbits can be assumed to lie in the equatorial plane ($\theta = \pi/2$).

1. **FALLING TOWARDS A BLACK HOLE**

Consider a falling object which starts at rest with respect to the spherical shell $r = r_0$.

(a) According to a faraway observer, how fast does the object appear to be falling as it crosses a smaller shell whose circumference is $2\pi R$?

(b) According to an observer on that smaller shell, how fast does the object appear to be falling as it goes past?

(c) Find the limiting values of these speeds as the object approaches the horizon.
What, if anything, do you think these results mean physically? You may want to think about what happens "at" the horizon. For instance, what is the limiting value of $\frac{dt}{d\tau}$? If you "stand still" "on" the horizon, what sort of trajectory are you following? How fast does this mean you are going?

(d) How long does it take the object to reach the horizon, as measured by its own clock? (OPTIONAL CHALLENGE: As measured from far away?)

The necessary integrals are hard! Therefore assume that $r_0 = \infty$ (the object started from rest at infinity), and find the time it takes to fall from $r = R$ to $r = 2m$. You may further assume $R = 4m$ if you wish.

2. **NO ESCAPE FROM A BLACK HOLE**

The goal of this problem is to establish an upper bound on the maximum proper time it takes to reach the singularity at $r = 0$ starting at the horizon $r = 2m$.

(a) Use the line element to show that *inside* the horizon the following relation holds:

$$\left|\frac{dr}{d\tau}\right| \geq \sqrt{\frac{2m}{r} - 1}$$

Assume that the Schwarzschild line element is valid inside the horizon, that is, for $r < 2m$, and recall that the proper time along any timelike trajectory is given by $d\tau^2 = -ds^2$.

(b) Assuming equality, find the elapsed proper time to travel from $r = 2m$ to $r = 0$; this is the desired upper bound.

You should feel free to use integral tables or computer algebra, but should document doing so. If you prefer to integrate by hand, the substitution $\sin\alpha = \sqrt{\frac{r}{2m}}$ may be helpful, and you may wish to recall that $\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$.

(c) How long does it take to reach the center of a solar mass black hole, starting at the horizon, if you do everything you can to resist falling in?