DIFFERENTIAL FORMS IN A NUTSHELL

($\wedge$, $*$, d, and all that)

1. WEDGE PRODUCTS

Differential forms are integrands, the things one integrates. So dx is a differential form (a *1-form*), and so is $dx\,dy$ (a *2-form*). However, orientation matters; think about change of variables, where for instance

$$du\,dv = \frac{\partial(u,v)}{\partial(x,y)}\,dx\,dy$$

One often puts an absolute value on the RHS, but that's misleading. For example, flux depends on the orientation of the surface, so it's a feature, not a bug, for integrals to depend on the ordering ("handedness") of the coordinates.

In order to emphasize that order matters, we write the *exterior product* of differential forms using the symbol $\wedge$, which is read as "wedge". The above formula becomes

$$du \wedge dv = \frac{\partial(u,v)}{\partial(x,y)}\,dx \wedge dy$$

from which we see that

$$dy \wedge dx = -dx \wedge dy$$
$$dx \wedge dx = 0$$

So order matters. But parentheses are not needed; the wedge product is associative (so $dx \wedge dy \wedge dz$ is unambiguous).

It is worth noting that, by construction, differential forms automatically incorporate change of variables. For example, comparing rectangular and polar coordinates, we have

$$dx \wedge dy = r\,dr \wedge d\phi$$

an equality which does not in fact hold for differentials, despite frequent usage to the contrary. (The correct statement in traditional language is that the corresponding *integrals* are equal.)

Given coordinates $\{x^i\}$ on an n-dimensional space M, it is easy to see that any 1-form can be written as

$$\alpha = \alpha_i\,dx^i$$

where we have adopted the *Einstein summation convention* in which a sum is implied by the presence of repeated indices. In other words, $\{dx^i\}$ forms a basis for 1-forms. Similarly, $\{dx^i \wedge dx^j\}$ with $i < j$ is a basis for 2-forms, and so forth. We also consider scalars (functions) to be 0-forms, and if necessary interpret wedge products of functions as ordinary multiplication.

For further insight into the meaning of $\wedge$, the reader is encouraged to work out $\alpha \wedge \beta$ for two arbitrary 1-forms in 3-dimensional Euclidean space.

2. ORTHONORMAL FRAMES

We will always work in an orthonormal basis. Intuitively, this means that the basis 1-forms measure "infinitesimal distance" in mutually orthogonal directions. Two such bases in 2-dimensional Euclidean space are $\{dx, dy\}$ and $\{dr, r\,d\phi\}$, which can easily be remembered using either the infinitesimal vector displacement

$$d\vec{\boldsymbol{r}} = dx\,\hat{\boldsymbol{\imath}} + dy\,\hat{\boldsymbol{\jmath}} = dr\,\hat{\boldsymbol{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}}$$

or the line element (also called the *metric*)

$$ds^2 = d\vec{\boldsymbol{r}} \cdot d\vec{\boldsymbol{r}} = dx^2 + dy^2 = dr^2 + r^2\,d\phi^2$$

In general, we will write our orthonormal basis of 1-forms as $\{\sigma^i\}$, so that a generic 1-form takes the form

$$\beta = \beta_i\,\sigma^i$$

We introduce the notation $g(\alpha, \beta)$ for the "dot product" of two 1-forms, which is easy to compute in terms of the multiplication table for an orthonormal basis. However, we do not require the dot product to be positive definite; in particular, the "squared magnitude" of the elements in our basis can be either positive or negative.

It is straightforward to extend the inner product g to higher-rank differential forms, although one must be careful with signs.

3. HODGE DUAL

Given an orthonormal basis in n dimensions, there are exactly two choices of *volume element*, that is, of a unit n-form, obtained by multiplying together the basis 1-forms in any order – two such products are the same if the orderings differ by an even permutation. Choosing one of these volume elements fixes an *orientation*. For example, in two Euclidean dimensions, the standard orientation is given by

$$\omega = dx \wedge dy = dr \wedge r\,d\phi$$

Given any p-form, there is a natural $(n-p)$-form associated with it, which is roughly the "missing pieces" needed to make up the (given) volume element ω. We write $*\alpha$ for the form associated in this way with α, which is called the *Hodge dual* of α. Thus, $*dx = dy$, but $*dy = -dx$. In general,

$$\alpha \wedge *\beta = g(\alpha, \beta)\,\omega$$

for any p-forms α, β, and this can be used to work out $*\beta$.

An important property of the Hodge dual is that

$$** = (-1)^{p(n-p)+s}$$

where p is the rank of the form being acted on, and s is the *signature* of the metric, which we take to be the number of minus signs.

For further insight into the meaning of $*$, the reader is encouraged to work out $*(\alpha \wedge *\beta)$ for two arbitrary 1-forms in 3-dimensional Euclidean space.

4. EXTERIOR DIFFERENTIATION

Not only do we integrate differential forms (in the obvious way); we also differentiate them. We already know how to differentiate 0-forms, namely

$$d(f) = df = \frac{\partial f}{\partial x^i}\, dx^i$$

We can generalize this operation to higher rank forms by requiring

$$d(f\, dx \wedge ... \wedge dy) = df \wedge dx \wedge ... \wedge dy$$

from which it follows that

$$\begin{aligned} d^2 &= 0 \\ d(\alpha \wedge \beta) &= d\alpha \wedge \beta + (-1)^p\, \alpha \wedge d\beta \end{aligned}$$

where α is a p-form.

For further insight into the meaning of d, the reader is encouraged to work out $*d\alpha$ and $*d{*}\beta$ for two arbitrary 1-forms in 3-dimensional Euclidean space.

5. CONNECTIONS & CURVATURE

The connection 1-forms describe how the basis changes. We always work with the *Levi-Civita* connection, the unique *metric-compatible*, *torsion-free* connection, which is the unique solution to the system of equations

$$\begin{aligned} d\sigma^i + \omega^i{}_j \wedge \sigma^j &= 0 \\ \omega_{ij} + \omega_{ji} &= 0 \end{aligned}$$

where the "up" and "down" indices incorporate a factor of -1 for "negatively normed" basis elements. (An equivalent construction can be given in terms of the corresponding vector basis. In this description, the extra signs in the second equation arise naturally in terms of the dot product.)

The curvature 2-forms describe the shape of the given space, and are defined by

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_m \wedge \omega^m{}_j$$

For further insight into the meaning of connections and curvature, the reader is encouraged to compute these forms in 2-dimensional Euclidean space in both rectangular and polar coordinates.