THE GEOMETRY OF SPECIAL RELATIVITY

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Lorentz transformations are just hyperbolic rotations.

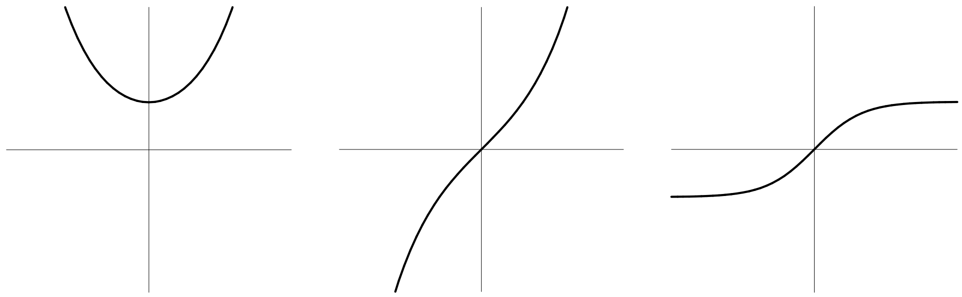

Figure 1: The graphs of $\cosh\beta$, $\sinh\beta$, and $\tanh\beta$, respectively.

1 Hyperbola Geometry

In which a 2-dimensional non-Euclidean geometry is constructed, which will turn out to be identical with special relativity.

1.1 Trigonometry

The hyperbolic trig functions are usually defined using the formulas

$$\cosh\beta = \frac{e^{\beta}+e^{-\beta}}{2} \tag{1}$$

$$\sinh\beta = \frac{e^{\beta}-e^{-\beta}}{2} \tag{2}$$

and then

$$\tanh\beta = \frac{\sinh\beta}{\cosh\beta} \tag{3}$$

and so on. We will discuss an alternative definition below. The graphs of these functions are shown in Figure 1.

It is straightforward to verify from these definitions that

$$\cosh^2\beta - \sinh^2\beta = 1 \tag{4}$$

$$\sinh(\alpha+\beta) = \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta \tag{5}$$

$$\cosh(\alpha+\beta) = \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta \tag{6}$$

$$\tanh(\alpha+\beta) = \frac{\tanh\alpha+\tanh\beta}{1+\tanh\alpha\tanh\beta} \tag{7}$$

$$\frac{d}{d\beta}\sinh\beta = \cosh\beta \tag{8}$$

$$\frac{d}{d\beta}\cosh\beta = \sinh\beta \tag{9}$$

These hyperbolic trig identities look very much like their ordinary trig counterparts (except for signs). This similarity derives from the fact that

$$\cosh\beta \equiv \cos(i\beta) \tag{10}$$
$$\sinh\beta \equiv -i\sin(i\beta) \tag{11}$$

1.2 Distance

Euclidean distance is based on the *unit circle*, the set of points which are unit distance from the origin. Hyperbola geometry is obtained simply by using a different distance function! Measure the "squared distance" of a point $B = (x, y)$ from the origin using the definition

$$\delta^2 = x^2 - y^2 \tag{12}$$

Then the unit "circle" becomes the unit hyperbola

$$x^2 - y^2 = 1 \tag{13}$$

and we further restrict ourselves to the branch with $x > 0$. If B is a point on this hyperbola, then we can *define* the hyperbolic angle β between the line from the origin to B and the (positive) x-axis to be the *Lorentzian* length [1] of the arc of the unit hyperbola between B and the point $(1, 0)$. We could then *define* the hyperbolic trig functions to be the coordinates (x, y) of B, that is

$$\cosh\beta = x \tag{14}$$
$$\sinh\beta = y \tag{15}$$

and a little work shows that this definition is exactly the same as the one above. [2] This construction is shown in Figure 2, which also shows another

[1] No, we haven't defined this. In Euclidean geometry, the the length of a curve is obtained by integrating ds along the curve, where $ds^2 = dx^2 + dy^2$. In a similar way, the Lorentzian length is obtained by integrating $d\sigma$, where $d\sigma^2 = dx^2 - dy^2$.

[2] Use $x^2 - y^2 = 1$ to compute

$$d\beta^2 \equiv d\sigma^2 = dy^2 - dx^2 = \frac{dx^2}{x^2 - 1} = \frac{dy^2}{y^2 + 1}$$

then take the square root of either expression and integrate. (The integrals are hard.) Finally, solve for x or y in terms of β, yielding (1) or (2), respectively.

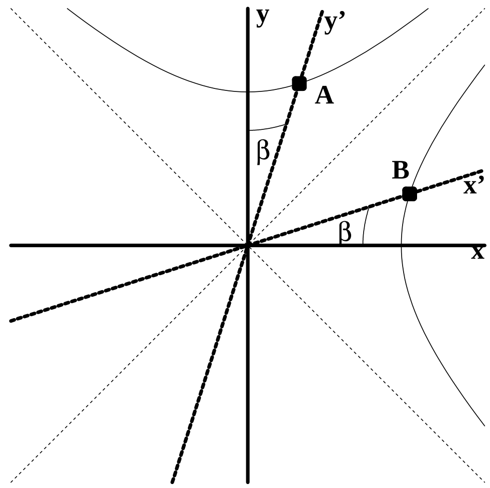

Figure 2: The unit hyperbola. The point A has coordinates $(\sinh\beta, \cosh\beta)$, and $B = (\cosh\beta, \sinh\beta)$.

"unit" hyperbola, given by $x^2 - y^2 = -1$. By symmetry, the point A on this hyperbola has coordinates $(x, y) = (\sinh\beta, \cosh\beta)$. We will discuss the importance of this hyerbola later.

Many of the features of the graphs shown in Figure 1 follow immediately from this definition of the hyperbolic trig functions in terms of coordinates along the unit hyperbola. Since the minimum value of x on this hyperbola is 1, we must have $\cosh\beta \geq 1$. As β approaches $\pm\infty$, x approaches ∞ and y approaches $\pm\infty$, which agrees with the asymptotic behavior of the graphs of $\cosh\beta$ and $\sinh\beta$, respectively. Finally, since the hyperbola has asymptotes $y = \pm x$, we see that $|\tanh\beta| < 1$, and that $\tanh\beta$ must approach ± 1 as β approaches $\pm\infty$.

So how do we measure the distance between two points? The "squared distance" was defined in (12), and can be positive, negative, or zero! We adopt the following convention: *Take the square root of the absolute value of the "squared distance".* As we will see in the next chapter, it will also be important to remember whether the "squared distance" was positive or negative, but this corresponds directly to whether the distance is "mostly horizontal" or "mostly vertical".

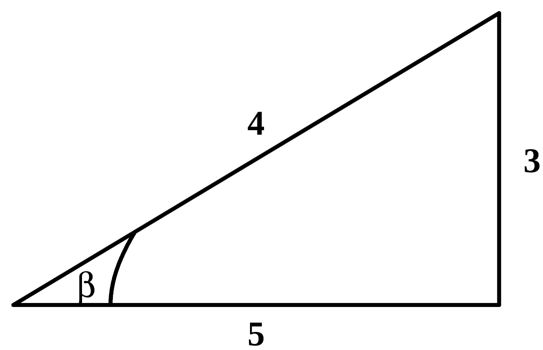

Figure 3: A hyperbolic triangle with $\tanh\beta = \frac{3}{5}$.

1.3 Triangle Trig

We now recast ordinary triangle trig into hyperbola geometry.

Suppose you know $\tanh\beta = \frac{3}{5}$, and you wish to determine $\cosh\beta$. One can of course do this algebraically, using the identity

$$\cosh^2\beta = \frac{1}{1 - \tanh^2\beta} \tag{16}$$

But it is easier to draw *any* triangle containing an angle whose hyperbolic tangent is $\frac{3}{5}$. In this case, the obvious choice would be the triangle shown in Figure 3, with sides of 3 and 5.

What is $\cosh\beta$? Well, we first need to work out the length δ of the hypotenuse. The (hyperbolic) Pythagorean Theorem tells us that

$$5^2 - 3^2 = \delta^2 \tag{17}$$

so δ is clearly 4. Take a good look at this 3-4-5 triangle of hyperbola geometry, which is shown in Figure 3! But now that we know all the sides of the triangle, it is easy to see that $\cosh\beta = \frac{5}{4}$.

Trigonometry is not merely about ratios of sides, it is also about projections. Another common use of triangle trig is to determine the sides of a triangle given the hypotenuse d and one angle β. The answer, of course, is that the sides are $d\cosh\beta$ and $d\sinh\beta$, as shown in in Figure 4.

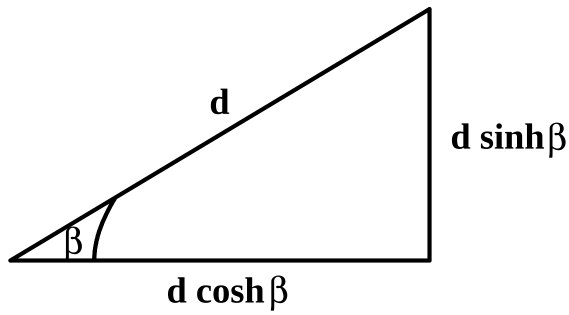

Figure 4: A hyperbolic triangle in which the hypotenuse and one angle are known.

1.4 Rotations

By analogy with the Euclidean case, we *define* a hyperbolic rotation through the relations

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \tag{18}$$

This corresponds to "rotating" both the x and y axes into the first quadrant, as shown in Figure 2. While this may seem peculiar, it is easily verified that the "distance" is invariant, that is,

$$x^2 - y^2 \equiv x'^2 - y'^2 \tag{19}$$

which follows immediately from the hyperbolic trig identity (4).

1.5 Projections

We can ask the same question as we did for Euclidean geometry. Consider a rectangle of width 1 whose sides are parallel to the unprimed axes. How wide is it when measured in the primed coordinates? It turns out that the width of the box in the primed coordinate system is *less than* 1. This is length contraction, to which we will return in the next section, along with time dilation.

1.6 Addition Formulas

What is the slope of the line from the origin to the point A in Figure 2? The equation of this line, the y'-axis, is

$$x = y \tanh \beta \tag{20}$$

Consider now a line with equation

$$x' = y' \tanh \alpha \tag{21}$$

What is its (unprimed) slope? Again, slopes don't add, but (hyperbolic) angles do; the answer is that

$$x = y \tanh(\alpha + \beta) \tag{22}$$

which can be expressed in terms of the slopes $\tanh \alpha$ and $\tanh \beta$ using (7). As discussed in more detail in the next section, this is the Einstein addition formula!

2 The Geometry of Special Relativity

In which it is shown that special relativity is just hyperbolic geometry.

2.1 Spacetime Diagrams

A brilliant aid in understanding special relativity is the *Surveyor's parable* introduced by Taylor and Wheeler [1, 2]. Suppose a town has daytime surveyors, who determine North and East with a compass, nighttime surveyors, who use the North Star. These notions of course differ, since magnetic north is not the direction to the North Pole. Suppose further that both groups measure north/south distances in miles and east/west distances in meters, with both being measured from the town center. How does one go about comparing the measurements of the two groups?

With our knowledge of Euclidean geometry, we see how to do this: Convert miles to meters (or vice versa). Furthermore, distances computed with the Pythagorean theorem do not depend on which group does the surveying. Finally, it is easily seen that "daytime coordinates" can be obtained from "nighttime coordinates" by a simple rotation. The moral of this parable is therefore:

1. *Use the same units.*
2. *The (squared) distance is invariant.*
3. *Different frames are related by rotations.*

Applying that lesson to relativity, the first thing to do is to measure both time and space in the same units. How does one measure distance in seconds? that's easy: simply multiply by c. Thus, since $c = 3 \times 10^8\ \frac{\mathrm{m}}{\mathrm{s}}$, 1 second of distance is just 3×10^8 m. [3] Note that this has the effect of setting $c = 1$, since the number of seconds (of distance) traveled by light in 1 second (of time) is precisely 1.

Of course, it is also possible to measure time in meters: simply divide by c. Thus, 1 meter of time is the time it takes for light (in vacuum) to travel 1 meter. Again, this has the effect of setting $c = 1$.

[3] A similar unit of distance is the *lightyear*, namely the distance traveled by light in 1 year, which would here be called simply a *year* of distance.

2.2 Lorentz Transformations

The Lorentz transformation between a frame (x, t) at rest and a frame (x', t') moving to the right at speed v was derived in class. The transformation from the moving frame to the frame at rest is given by

$$\begin{aligned} x &= \gamma\,(x' + vt') \qquad (23)\\ t &= \gamma\left(t' + \frac{v}{c^2}x'\right) \qquad (24)\end{aligned}$$

where

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (25)$$

The key to converting this to hyperbola geometry is to measure space and time in the same units by replacing t by ct. The transformation from the moving frame, which we now denote (x', ct'), to the frame at rest, now denoted (x, ct), is given by

$$\begin{aligned} x &= \gamma\,(x' + \frac{v}{c}\,ct') \qquad (26)\\ ct &= \gamma\left(ct' + \frac{v}{c}\,x'\right) \qquad (27)\end{aligned}$$

which makes the symmetry between these equations much more obvious.

We can simplify things still further. Introduce the *rapidity* β via [4]

$$\frac{v}{c} = \tanh\beta \qquad (28)$$

Inserting this into the expression for γ we obtain

$$\gamma = \frac{1}{\sqrt{1-\tanh^2\beta}} = \sqrt{\frac{\cosh^2\beta}{\cosh^2\beta - \sinh^2\beta}} = \cosh\beta \qquad (29)$$

and

$$\frac{v}{c}\,\gamma = \tanh\beta\cosh\beta = \sinh\beta \qquad (30)$$

[4]WARNING: Some authors use β for $\frac{v}{c}$, not the rapidity.

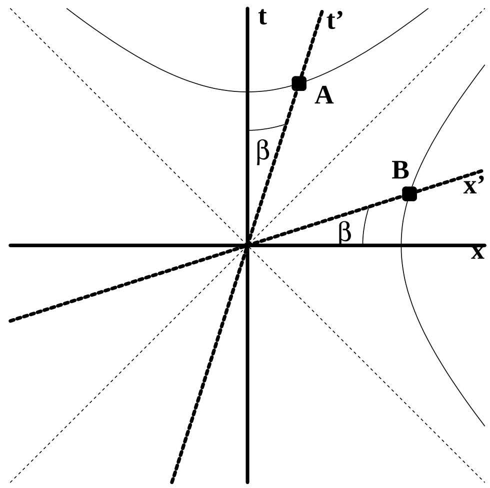

Figure 5: The Lorentz transformation between an observer at rest and an observer moving at speed $\frac{v}{c} = \tanh\beta$ is shown as a hyperbolic rotation. The point A has coordinates $(\sinh\beta, \cosh\beta)$, and $B = (\cosh\beta, \sinh\beta)$. (Units have been chosen such that $c = 1$.)

Inserting these identities into the Lorentz transformations above brings them to the remarkably simple form

$$x = x' \cosh\beta + ct' \sinh\beta \tag{31}$$

$$ct = x' \sinh\beta + ct' \cosh\beta \tag{32}$$

which in matrix form are just

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix} \tag{33}$$

But (33) is just (18), with $y = ct$!

Thus, Lorentz transformations are just hyperbolic rotations! As noted in the previous section, the invariance of the interval follows immediately from the fundamental hyperbolic trig identity (4). This invariance now takes the form

$$x^2 - c^2t^2 \equiv x'^2 - c^2t'^2 \tag{34}$$

We thus have precisely the situation described in Figure 2, but with y replaced by ct; this is shown in Figure 5.

2.3 Space and Time

We now return to the peculiar fact that the "squared distance" between two points can be positive, negative, or zero. This sign is positive for horizontal distances and negative for vertical distances. But these directions correspond to the coordinates x and t, and measure space and time, respectively — as seen by the given observer. But *any* observer's space axis must intersect the unit hyperbola somewhere, and hence corresponds to positive "squared distance". Such directions have more space than time, and will be called *spacelike*. Similarly, any observer's time axis intersects the hyperbola $x^2 - c^2t^2 = -1$, corresponding to negative "squared distance"; such directions are *timelike*.

What about diagonal lines at a (Euclidean!) angle of 45°? These correspond to a "squared distance" of zero — and to moving at the speed of light. All observers agree about these directions, which will be called *lightlike*. In hyperbola geometry, there are thus preferred directions of "length zero". Indeed, this is the geometric realization of the idea that the speed of light is the same for all observers!

It is important to realize that *every* spacelike direction corresponds to the space axis for *some* observer. Events separated by a spacelike line occur at the simultaneously for that observer — and the (square root of the) "squared distance" is just the distance between the events as seen by that observer. Similarly, events separated by a timelike line occur at the same place for some observer, and the (square root of -1 times the) "squared distance" is just the time which elapses between the events as seen by that observer.

On the other hand, events separated by a timelike line do not occur simultaneously for *any* observer! We can thus divide the spacetime diagram into causal regions as follows: Those points connected to the origin by spacelike lines occur "now" for some observer, whereas those points connected to the origin by timelike lines occur unambiguously in the future or the past. This is shown in Figure 6. [5]

In order to be able to make sense of cause and effect, only events in our past can influence us, and we can only influence events in our future. Put differently, if information could travel faster than the speed of light, then different observers would no longer be able to agree on cause and effect.

[5] With two or more spatial dimensions, the lightlike directions would form a surface called the *light cone*, and the regions labeled "now" would be connected.

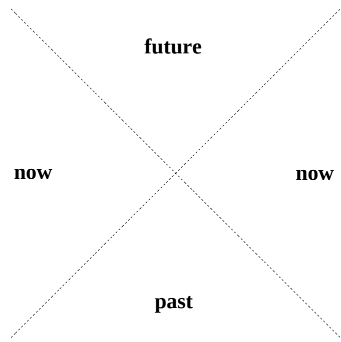

Figure 6: The causal relationship between points in spacetime and the origin.

2.4 Dot Product

In Euclidean geometry, distances can be described by taking the (squared!) length of a vector using the dot product. Denoting the unit vectors in the x and y directions by $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$, respectively, then the vector from the origin to the point (x, y) is just

$$\vec{\boldsymbol{r}} = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} \tag{35}$$

whose (squared) length is just

$$|\vec{\boldsymbol{r}}|^2 = \vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{r}} = x^2 + y^2 \tag{36}$$

It is straightforward to generalize this to hyperbola geometry. Denote the unit vectors in the t and x directions by $\hat{\boldsymbol{t}}$ and $\hat{\boldsymbol{x}}$. [6] Then the (Lorentzian) dot product can be defined by the requirement that this be an orthonormal basis, in the sense that

$$\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{x}} = 1 \tag{37}$$

$$\hat{\boldsymbol{t}} \cdot \hat{\boldsymbol{t}} = -1 \tag{38}$$

$$\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{t}} = 0 \tag{39}$$

Any point $(x,\, ct)$ in spacetime can thus be identified with the vector

$$\vec{\boldsymbol{r}} = x\,\hat{\boldsymbol{x}} + ct\,\hat{\boldsymbol{t}} \tag{40}$$

[6] Unit vectors are dimensionless! It is neither necessary nor desirable to include a factor of c in the definition of $\hat{\boldsymbol{t}}$.

from the origin to that point, whose "squared length" is just the "squared distance" from the origin, namely

$$|\vec{\boldsymbol{r}}|^2 = \vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{r}} = x^2 - c^2t^2 \tag{41}$$

One of the fundamental properties of the Euclidean dot product is that

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = |\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}| \cos\theta \tag{42}$$

where θ is the (smallest) angle between the directions of $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$. This relationship between the dot product and projections of one vector along another can in fact be used to *define* the dot product. What happens in hyperbola geometry?

First of all, the dot product can be used to define right angles: Two vectors $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ are said to be *orthogonal* (or *perpendicular*) precisely when their dot product is zero, that is

$$\vec{\boldsymbol{u}} \perp \vec{\boldsymbol{v}} \Longleftrightarrow \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = 0 \tag{43}$$

We will adopt this definition unchanged in hyperbola geometry.

When are $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ perpendicular? Assume first that $\vec{\boldsymbol{u}}$ is spacelike. We can assume without loss of generality that $\vec{\boldsymbol{u}}$ is a unit vector, in which case it takes the form

$$\vec{\boldsymbol{u}} = \cosh\alpha\,\hat{\boldsymbol{x}} + \sinh\alpha\,\hat{\boldsymbol{t}} \tag{44}$$

What vectors are perpendicular to $\vec{\boldsymbol{u}}$? One such vector is

$$\vec{\boldsymbol{v}} = \sinh\alpha\,\hat{\boldsymbol{x}} + \cosh\alpha\,\hat{\boldsymbol{t}} \tag{45}$$

and it is easy to check that all other solutions are multiples of this one. Note that $\vec{\boldsymbol{v}}$ is timelike! Had we assumed instead that $\vec{\boldsymbol{v}}$ were timelike, we would merely have interchanged the roles of $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$.

Furthermore, $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ are just the space and time axes, respectively, of an observer moving with speed $\frac{v}{c} = \tanh\alpha$. So orthogonal directions correspond precisely to the coordinate axes of some observer.

What if $\vec{\boldsymbol{u}}$ is lightlike? It is a peculiarity of Lorentzian (hyperbola) geometry that there are nonzero vectors of length zero. But since the dot product gives the length, having length zero means that lightlike vectors are perpendicular to themselves!

We can finally define the *length* of a vector $\vec{\boldsymbol{v}}$ by

$$|\vec{\boldsymbol{v}}| = \sqrt{|\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}|} \tag{46}$$

If $\vec{\boldsymbol{v}}$ is spacelike we can write

$$\vec{\boldsymbol{v}} = |\vec{\boldsymbol{v}}|(\cosh\alpha\,\hat{\boldsymbol{x}} + \sinh\alpha\,\hat{\boldsymbol{t}}) \tag{47}$$

while if $\vec{\boldsymbol{v}}$ is timelike we can write

$$\vec{\boldsymbol{v}} = |\vec{\boldsymbol{v}}|(\sinh\alpha\,\hat{\boldsymbol{x}} + \cosh\alpha\,\hat{\boldsymbol{t}}) \tag{48}$$

(If $\vec{\boldsymbol{v}}$ is lightlike, $|\vec{\boldsymbol{v}}| = 0$, so no such expression exists.)

The above argument shows that timelike vectors can only be perpendicular to spacelike vectors, and *vice versa.* We will also say in this case that the vectors form a *right angle.* Recall that hyperbolic angles were defined along the unit hyperbola, hence only exist (as originally defined) between spacelike directions! It is straightforward to extend this to timelike directions using the hyperbola $x^2 - ct^2 = -1$; this was implicitly done when drawing Figure 2. But there is no hyperbola relating timelike directions to spacelike ones. Thus, a "right angle" isn't an angle at all!

A *right triangle* is one which contains a right angle. By the above discussion, one of the legs of such a triangle must be spacelike, and the other timelike. Consider first the case where the hypotenuse is either spacelike or timelike. The only hyperbolic angle in such a triangle is the one between the hypotenuse and the leg of the same type, that is between the two timelike sides if the hypotenuse is timelike, and between the two spacelike sides if the hypotenuse is spacelike. Several such hyperbolic right triangles are shown in Figures 7. It is also possible for the hypotenuse to be null, as shown in Figure 8. Such triangles do not have any hyperbolic angles!

What happens if we take the dot product between two spacelike vectors? We can assume without loss of generality that one vector is parallel to the x axis, in which case we have

$$\vec{\boldsymbol{u}} = |\vec{\boldsymbol{u}}|\,\hat{\boldsymbol{x}} \tag{49}$$

$$\vec{\boldsymbol{v}} = |\vec{\boldsymbol{v}}|(\cosh\alpha\,\hat{\boldsymbol{x}} + \sinh\alpha\,\hat{\boldsymbol{t}}) \tag{50}$$

so that the dot product satisfies

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = |\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}|\cosh\alpha \tag{51}$$

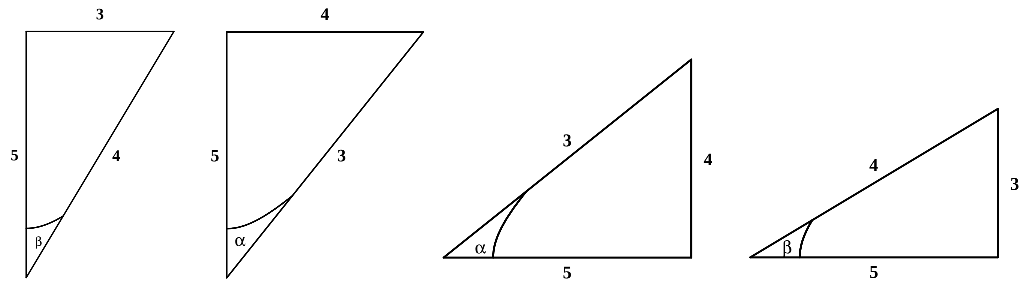

Figure 7: Some hyperbolic right triangles.

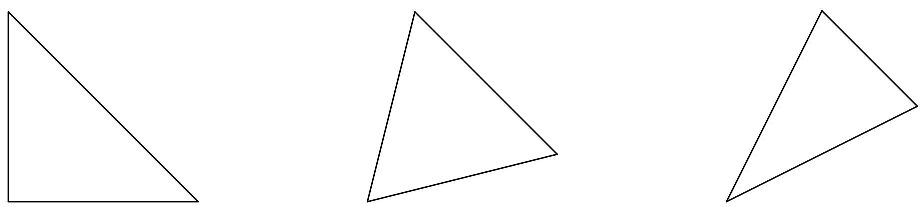

Figure 8: More hyperbolic right triangles. The right angle is on the left!

What happens if both vectors are timelike? The above argument still works, except that the roles of $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{t}}$ must be interchanged, resulting in

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = -|\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}| \cosh \alpha \tag{52}$$

In both cases, note that $|\vec{\boldsymbol{v}}| \cosh \alpha$ is the projection of $\vec{\boldsymbol{v}}$ along $\vec{\boldsymbol{u}}$; see Figure 9.

But what happens if we take the dot product between a timelike vector and a spacelike vector? We can again assume without loss of generality that

Figure 9: Hyperbolic projections between two spacelike vectors, or between two timelike vectors.

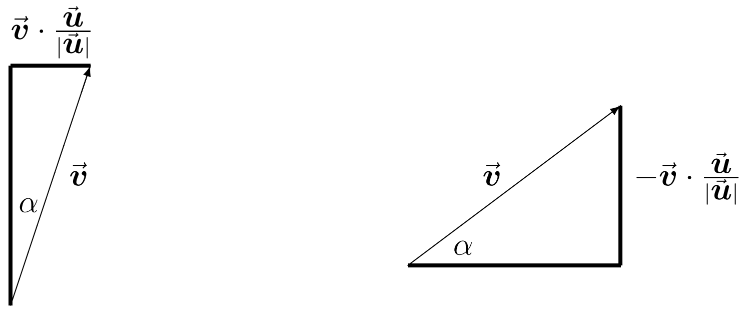

Figure 10: Hyperbolic projections between timelike and spacelike vectors.

the spacelike vector is parallel to the x axis, so that

$$\vec{\boldsymbol{u}} = |\vec{\boldsymbol{u}}|\,\hat{\boldsymbol{x}} \tag{53}$$

$$\vec{\boldsymbol{v}} = |\vec{\boldsymbol{v}}|(\sinh\alpha\,\hat{\boldsymbol{x}} + \cosh\alpha\,\hat{\boldsymbol{t}}) \tag{54}$$

The dot product now satisfies

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = |\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}|\sinh\alpha \tag{55}$$

At first sight, this is something new. But note from the first drawing in Figure 10 that $\vec{\boldsymbol{v}}\sinh\alpha$ is just the projection of $\vec{\boldsymbol{v}}$ along $\vec{\boldsymbol{u}}$! The new feature here is that we can't define the angle between a timelike direction and a spacelike direction. The only angle in the triangle which is defined is the one shown! [7]

[7] Alternatively, we could have assumed that the timelike vector was parallel to the t axis, resulting in the second drawing in Figure 10. The conclusion is the same, although now it represents the projection of $\vec{\boldsymbol{u}}$ along $\vec{\boldsymbol{v}}$.

3 Applications

3.1 Addition of Velocities

What is the rapidity β? Consider an observer moving at speed v to the right. This observer's world line intersects the unit hyperbola

$$c^2t^2 - x^2 = 1 \qquad (ct > 0) \tag{56}$$

at the point $A = (\sinh\beta, \cosh\beta)$; this line has "slope" [8]

$$\frac{v}{c} = \tanh\beta \tag{57}$$

as required. Thus, β can be thought of as the *hyperbolic angle* between the ct-axis and the worldline of a moving object. As discussed in the preceding section, β turns out to be precisely the distance from the axis as measured along the hyperbola (in hyperbola geometry!). This was illustrated in Figure 5.

Consider therefore an object moving at speed u relative to an observer moving at speed v. Their rapidities are given by

$$\frac{u}{c} = \tanh\alpha \tag{58}$$

$$\frac{v}{c} = \tanh\beta \tag{59}$$

To determine the resulting speed with respect to an observer at rest, simply add the *rapidities*! One way to think of this is that you are adding the arc lengths along the hyperbola. Another is that you are following a (hyperbolic) rotation through a (hyperbolic) angle β (to get to the moving observer's frame) with a rotation through an angle α. In any case, the resulting speed w is given by

$$\frac{w}{c} = \tanh(\alpha+\beta) = \frac{\tanh\alpha + \tanh\beta}{1 + \tanh\alpha\tanh\beta} = \frac{\frac{u}{c} + \frac{v}{c}}{1 + \frac{uv}{c^2}} \tag{60}$$

which is — finally — precisely the Einstein addition formula!

3.2 Length Contraction

We now return to the question of how "wide" things are.

[8] It is not obvious whether "slope" should be defined by $\frac{\Delta x}{c\,\Delta t}$ or by the reciprocal of this expression. This is further complicated by the fact that both (x, ct) and (ct, x) are commonly used to denote the coordinates of the point A!

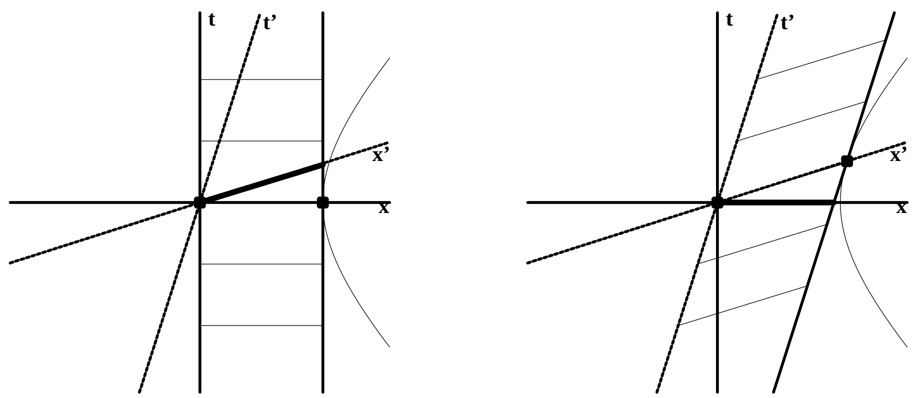

Figure 11: Length contraction as a hyperbolic projection.

Consider first a meter stick at rest. In spacetime, the stick "moves" vertically, that is, it ages. This situation is shown in the first sketch in Figure 11, where the horizontal lines show the meter stick at various times (according to an observer at rest). How "wide" is the *worldsheet* of the stick? The observer at rest of course measures the length of the stick by locating both ends *at the same time*, and measuring the distance between them. At $t = 0$, this corresponds to the 2 heavy dots in the sketch, one at the origin and the other on the unit hyperbola. But *all* points on the unit hyperbola are at an interval of 1 meter from the origin. The observer at rest therefore concludes, unsurprisingly, that the meter stick is 1 meter long.

How long does a moving observer think the stick is? This is just the "width" of the worldsheet *as measured by the moving observer*. This observer follows the same procedure, by locating both ends of the stick *at the same time*, and measuring the distance between them. But time now corresponds to t', not t. At $t' = 0$, this measurement corresponds to the heavy line in the sketch. Since this line fails to reach the unit hyperbola, it is clear that the moving observer measures the length of a stationary meter stick to be less than 1 meter. This is length contraction.

To determine the exact value measured by the moving observer, compute the intersection of the line $x = 1$ (the right-hand edge of the meter stick) with the line $t' = 0$ (the x'-axis), or equivalently $ct = x \tanh \beta$, to find that

$$ct = \tanh \beta \tag{61}$$

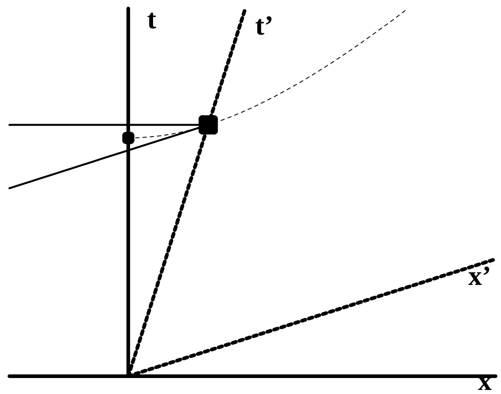

Figure 12: Time dilation as a hyperbolic projection.

so that x' is just the interval from this point to the origin, which is

$$x' = \sqrt{x^2 - c^2t^2} = \sqrt{1 - \tanh^2\beta} = \frac{1}{\cosh\beta} \tag{62}$$

What if the stick is moving and the observer is at rest? This situation is shown in the second sketch in Figure 11. The worldsheet now corresponds to a "rotated rectangle", indicated by the parallelograms in the sketch. The fact that the meter stick is 1 meter long in the moving frame is shown by the distance between the 2 heavy dots (along $t' = 0$), and the measurement by the observer at rest is indicated by the heavy line (along $t = 0$). Again, it is clear that the stick appears to have shrunk, since the heavy line fails to reach the unit hyperbola.

Thus, a moving object appears shorter by a factor $1/\cosh\beta$. It doesn't matter whether the stick is moving, or the observer; all that matters is their relative motion.

3.3 Time Dilation

We now investigate moving clocks. Consider first the smaller dot in Figure 12. This corresponds to $ct = 1$ (and $x = 0$), as evidenced by the fact that this point is on the (other) unit hyperbola, as shown. Similarly, the larger dot, lying on the same hyperbola, corresponds to $ct' = 1$ (and $x' = 0$). The horizontal line emanating from this dot gives the value of ct there, which is clearly greater than 1. This is the time measured by the observer at rest when the moving clock says 1; the moving clock therefore runs slow. But now consider the diagonal line emanating from the larger dot. At all points along this line, $ct' = 1$. In particular, at the smaller dot we must have $ct' > 1$. This is the time measured by the moving observer when the clock at rest says 1; the moving observer therefore concludes the clock at rest runs slow!

There is no contradiction here; one must simply be careful to ask the right question. In each case, observing a clock in another frame of reference corresponds to a projection. In each case, a clock in relative motion to the observer appears to run slow.

3.4 Doppler Shift

The frequency f of a beam of light is related to its wavelength λ by the formula

$$f\lambda = c \tag{63}$$

How do these quantities depend on the observer?

Consider an inertial observer moving to the right in the laboratory frame who is carrying a flashlight that is pointing to the left; see Figure 13. Then the moving observer is traveling along a path of the form $x' = x'_1 = \text{const}$. Suppose the moving observer turns on the flashlight (at time t'_1) just long enough to emit 1 complete wavelength of light, and that this takes time dt'. Then the moving observer "sees" a wavelength

$$\lambda' = c\,dt' \tag{64}$$

According to the lab, the flashlight was turned on at the event (t_1, x_1), and turned off dt_1 seconds later, during which time the moving observer moved a distance dx_1 meters to the right. But when was the light received, at $x = 0$, say?

Let $(t_0, 0)$ denote the first reception of light by a lab observer at $x = 0$, and suppose this observer sees the light stay on for dt_0 seconds. Since light travels at the speed of light, we have the equations

$$\begin{aligned} c(t_0 - t_1) &= x_1 && (65) \\ c[(t_0 + dt_0) - (t_1 + dt_1)] &= x_1 + dx_1 && (66) \end{aligned}$$

from which it follows that

$$c(dt_0 - dt_1) = dx_1 \tag{67}$$

so that

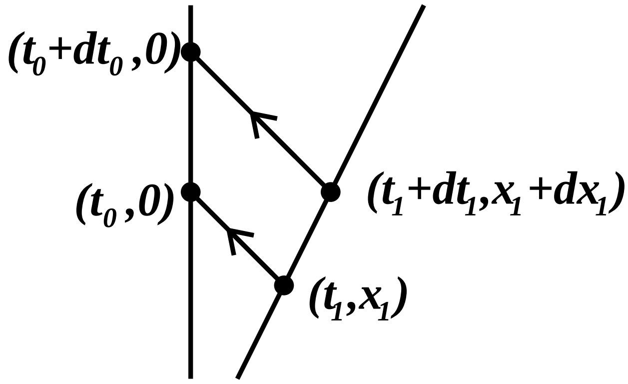

Figure 13: The Doppler effect: An observer moving to the right emits a pulse of light to the left, which is later seen by a stationary observer. The wavelengths measured by the two observers differ, causing a *Doppler shift* in the frequency.

$$
\begin{aligned}
c\,dt_0 &= dx_1 + c\,dt_1 && (68)\\
&= (dx_1' \cosh\beta + c\,dt_1' \sinh\beta) + (c\,dt_1' \cosh\beta + dx_1' \sinh\beta) && (69)\\
&= (\cosh\beta + \sinh\beta)\,c\,dt_1' && (70)
\end{aligned}
$$

since $dx_1' = 0$. But the wavelength as seen in the lab is

$$\lambda = c\,dt_0 \tag{71}$$

so that

$$
\begin{aligned}
\frac{\lambda}{\lambda'} &= \frac{dt_0}{dt_1'} = \cosh\beta + \sinh\beta \\
&= \cosh\beta\,(1+\tanh\beta) = \gamma\left(1+\frac{v}{c}\right) = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} && (72)
\end{aligned}
$$

The frequencies transform inversely, that is

$$\frac{f'}{f} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \tag{73}$$