CONNECTIONS

1. THEORY

Suppose we are given the infinitesimal displacement in the form

$$d\vec{\boldsymbol{r}} = \sigma^i \hat{e}_i$$

where $\{\hat{e}_i\}$ is an orthonormal vector basis and the σ^i are 1-forms; $d\vec{\boldsymbol{r}}$ is thus a *vector-valued 1-form.* We note first of all that $d\vec{\boldsymbol{r}}$ provides a map from vectors to 1-forms, given by

$$v := \vec{\boldsymbol{v}} \cdot d\vec{\boldsymbol{r}}$$

We think of v and $\vec{\boldsymbol{v}}$ as physically equivalent, so we impose the condition

$$|v| = |\vec{\boldsymbol{v}}|$$

which leads by polarization to an inner product on 1-forms given by

$$g(\vec{\boldsymbol{v}} \cdot d\vec{\boldsymbol{r}}, \vec{\boldsymbol{w}} \cdot d\vec{\boldsymbol{r}}) := \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}$$

Not surprisingly, $\{\sigma^i\}$ is an orthonormal 1-form basis under this inner product. Furthermore, the inverse isomorphism from 1-forms to vectors is given by

$$\vec{\boldsymbol{v}} = g(v, d\vec{\boldsymbol{r}})$$

Finally, we note that $\{\sigma^i\}$ is just the dual basis to $\{\hat{e}_i\}$, under the action

$$v(\vec{\boldsymbol{w}}) := \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}$$

with v and $\vec{\boldsymbol{v}}$ related as above.

We wish to extend the exterior derivative operator on differential forms to (ordinary) vectors. We define

$$\omega_{ij} := \hat{e}_i \cdot d\hat{e}_j$$

and the goal is now to determine the *connection 1-forms* ω_{ij}. We first impose the condition that d be *metric compatible*, that is, that

$$d(\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}) = d\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} + \vec{\boldsymbol{v}} \cdot d\vec{\boldsymbol{w}}$$

from which it follows immediately that

$$0 = d(\hat{e}_i \cdot \hat{e}_j) = d\hat{e}_i \cdot \hat{e}_j + d\hat{e}_j \cdot \hat{e}_i$$

or in other words

$$\omega_{ji} + \omega_{ij} = 0$$

We also require that d be *torsion free*, which means by definition that [1]

$$d(d\vec{\boldsymbol{r}}) = 0$$

[1] This condition does *not* automatically follow from the requirement that $d^2 = 0$ on differential forms, since it is not obvious that $d\vec{\boldsymbol{r}}$ is d of anything; the position vector $\vec{\boldsymbol{r}}$ may not be in our vector space.

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Working out this condition, we have

$$d(d\vec{\boldsymbol{r}}) = d\sigma^i \hat{e}_i - \sigma^i \wedge d\hat{e}_i$$

so that

$$d(d\vec{\boldsymbol{r}}) \cdot \hat{e}_j = d\sigma^i \hat{e}_i \cdot \hat{e}_j - \sigma^i \wedge \omega_{ji}$$

Noting that $\hat{e}_i \cdot \hat{e}_j$ is constant (and in fact 0 or ± 1), we define

$$\sigma_j = \sigma^i \hat{e}_i \cdot \hat{e}_j$$

and the torsion-free condition becomes

$$d\sigma_j = -\omega_{ji} \wedge \sigma^i$$

Defining the components of ω_{ij}, known as *Christoffel symbols*, via

$$\omega_{ij} =: \mathit{\Gamma}_{ijk} \sigma^k$$

the metric-compatability condition becomes

$$\mathit{\Gamma}_{ijk} + \mathit{\Gamma}_{jik} = 0$$

Introducing the action of 2-forms on vectors via

$$(\alpha \wedge \beta)(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) := g(\alpha \wedge \beta, v \wedge w) = \alpha(\vec{\boldsymbol{v}})\, \beta(\vec{\boldsymbol{w}}) - \alpha(\vec{\boldsymbol{w}})\, \beta(\vec{\boldsymbol{v}})$$

we can rewrite the torsion-free condition as

$$d\sigma_j(\hat{e}_k, \hat{e}_\ell) = \mathit{\Gamma}_{jmn}(\sigma^m \wedge \sigma^n)(\hat{e}_k, \hat{e}_\ell) = \mathit{\Gamma}_{jk\ell} - \mathit{\Gamma}_{j\ell k}$$

Putting this all together, and taking a non-obvious combination of terms, we obtain

$$\begin{aligned} d\sigma_j(\hat{e}_k, \hat{e}_\ell) + d\sigma_k(\hat{e}_\ell, \hat{e}_j) - d\sigma_\ell(\hat{e}_j, \hat{e}_k) &= \mathit{\Gamma}_{jk\ell} - \mathit{\Gamma}_{j\ell k} + \mathit{\Gamma}_{k\ell j} - \mathit{\Gamma}_{kj\ell} - \mathit{\Gamma}_{\ell jk} + \mathit{\Gamma}_{\ell kj} \\ &= \mathit{\Gamma}_{jk\ell} - \mathit{\Gamma}_{j\ell k} + \mathit{\Gamma}_{k\ell j} + \mathit{\Gamma}_{jk\ell} + \mathit{\Gamma}_{j\ell k} - \mathit{\Gamma}_{k\ell j} \\ &= 2\mathit{\Gamma}_{jk\ell} \end{aligned}$$

from which the Christoffel symbols, and hence the connection 1-forms, are uniquely determined.

We have therefore proved that there is a unique torsion-free, metric-compatible connection, which is known as the *Levi-Civita connection*. The above formula for the connection components is a special case of the *Koszul formula* for the Levi-Civita connection.

2. PRACTICE

The Koszul formula is rarely the most efficient way to determine the connection. Armed with the above existence and uniqueness result, it is usually simpler to guess a solution of the equations [2]

$$\begin{aligned} d\sigma_j + \omega_{jk} \wedge \sigma^k &= 0 \\ \omega_{jk} + \omega_{kj} &= 0 \end{aligned}$$

If it works, you're done!

[2] If the signature is not zero, care must be taken to remember that $\sigma_j = \pm\sigma^j$.