The Geodesic Equation

What makes a straight line straight?

1   Flat Space

1.1   Rectangular Coordinates

What is the equation of a straight line? How about $y = mx + b$? In the (Euclidean) plane, a more general equation for straight lines is

$$ax + by = c \tag{1}$$

with a, b, c constant. What are the key properties of straight lines?

- *Straight lines don't bend;*

- *Straight lines give the shortest distance between two points.*

What does it mean to be straight? That the direction of the line does not change. In other words, the unit tangent vector to the curve remains constant.

Note that the *speed* does not need to be constant; only the direction remains the same. However, if we are only interested in the *shape* of the line, and not in the particular manner in which we move along it, it is convenient to use a description in which the velocity itself is constant. One way to do this is to use arclength itself as the parameter along the curve; such parameterizations are also called *unit speed.*

In two dimensions, then, we can describe any curve by its position (x, y), with the coordinates x and y being thought of as functions of arclength s

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along the curve. The unit tangent vector to the curve is then

$$\hat{\boldsymbol{T}} = \dot{x}\,\hat{\boldsymbol{\imath}} + \dot{y}\,\hat{\boldsymbol{\jmath}} \tag{2}$$

where we have used a dot to denote derivatives with respect to s. The condition that the curve be straight is then that the acceleration vanish, or equivalently that

$$\ddot{x} = 0 = \ddot{y} \tag{3}$$

1.2 Polar Coordinates

What does a straight line look like in polar coordinates? Radial lines are easy; simply set $\phi = $ constant, or better yet set $\tan \phi = $ constant (to get the entire line). But what about lines which do not go through the origin?

We already know the answer! Inserting the standard relation between polar and rectangular coordinates into Equation (1) immediately gives us

$$ar \cos \phi + br \sin \phi = c \tag{4}$$

But that's cheating; what if we couldn't use rectangular coordinates?

Our first characterization of straight lines was that they do not bend, or equivalently that they do not accelerate. The standard approach to the notion of straight lines in arbitrary spaces is in fact to generalize this lack of acceleration, thus showing that the (unit) tangent vector does not change. However, this approach requires knowing how to differentiate vectors in arbitrary spaces. We choose to avoid this extra machinery, the *covariant derivative*, and seek a different path.

We also characterized straight lines as representing the shortest distance between two points. We now explore this interpretation a bit further.

Suppose two points A and B, and a curve C between them, are given. If C is straight, it should also be the shortest path between A and B. So if we slightly perturb C (keeping the endpoints fixed), the length must increase. This leads us to the *calculus of variations*.

Suppose the original curve C is given by (r, ϕ), again thought of as a function of arclength s. Then the length of the curve is

$$s \,\Big|_A^B = \int_A^B ds \tag{5}$$

and we know that

$$ds^2 = dr^2 + r^2 d\phi^2 \tag{6}$$

in polar coordinates. Now consider a nearby curve, given by $(r + \delta r, \phi + \delta\phi)$, with δr and $\delta\phi$ (and their derivatives) suitably small. The arclength of this nearby curve can now be computed as follows:

$$
\begin{aligned}
s \Big|_A^B + \delta s &= \int_A^B \sqrt{d(r + \delta r)^2 + (r + \delta r)^2 \, d(\phi + \delta\phi)^2} \tag{7} \\
&\approx \int_A^B \sqrt{dr^2 + r^2 d\phi^2 + 2\, dr\, d\delta r + 2r\, \delta r\, d\phi^2 + 2r^2 d\phi\, d\delta\phi} \\
&= \int_A^B \sqrt{1 + 2\frac{dr}{ds}\frac{d}{ds}\delta r + 2r\, \delta r \left(\frac{d\phi}{ds}\right)^2 + 2r^2 \frac{d\phi}{ds}\frac{d}{ds}\delta\phi} \;\; ds \\
&\approx \int_A^B \left(1 + \frac{dr}{ds}\frac{d}{ds}\delta r + r\, \delta r \left(\frac{d\phi}{ds}\right)^2 + r^2 \frac{d\phi}{ds}\frac{d}{ds}\delta\phi\right) ds \\
&= \left(s + \frac{dr}{ds}\delta r + r^2 \frac{d\phi}{ds}\,\delta\phi\right)\Big|_A^B \\
&\quad + \int_A^B \left[\left(-\frac{d^2 r}{ds^2} + r\left(\frac{d\phi}{ds}\right)^2\right)\delta r - \frac{d}{ds}\left(r^2 \frac{d\phi}{ds}\right)\delta\phi\right] ds
\end{aligned}
$$

where we have discarded terms quadratic in δr and $\delta\phi$ (and their derivatives) in both approximations, and integrated by parts in the last step. Note however that δr and $\delta\phi$ are zero at the endpoints, which we held fixed. Comparing the first and last expressions yields

$$\delta s = \int_A^B \left[\left(-\ddot{r} + r\dot{\phi}^2\right)\delta r - \left(r^2 \dot{\phi}\right)^{\cdot} \delta\phi\right] ds \tag{8}$$

where we have again used a dot to denote derivatives with respect to s.

But if C is supposed to be the shortest path between A and B, then δs must be zero, for any (reasonable) choices of δr and $\delta\phi$! This is the variational principle, which is entirely analogous to the vanishing of the derivative of a function at a maximum or a minimum. We therefore conclude that C is

straight if and only if

$$\ddot{r} = r\dot{\phi}^2 \tag{9}$$
$$r^2\dot{\phi} = \text{constant} \tag{10}$$

To see that these equations are reasonable, we check that they are satisfied by our known solution (4). Differentiating both sides of (4) yields

$$(a\cos\phi + b\sin\phi)\dot{r} + (-a\sin\phi + b\cos\phi)r\dot{\phi} = 0 \tag{11}$$

which leads to

$$(a\cos\phi + b\sin\phi)^2\,\dot{r}^2 = (-a\sin\phi + b\cos\phi)^2\,r^2\,\dot{\phi}^2 \tag{12}$$

However, since we are using arclength, we know that

$$\dot{r}^2 + r^2\,\dot{\phi}^2 = \left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2 = 1 \tag{13}$$

Solving (13) for $\dot{r}$, inserting the result into (12), and expanding both sides yields

$$r^4\dot{\phi}^2 = \frac{c^2}{a^2 + b^2} \tag{14}$$

which agrees with (10). (Equation (9) follows from (10) using (13).)

2 Curved Space

2.1 General Case

We can repeat the above computation for any line element. Suppose that

$$ds^2 = g_{ab}\,dx^a\,dx^b \tag{15}$$

Letting $x^a \mapsto x^a + \delta x^a$ leads to

$$s\,\Big|_A^B + \delta s = \int_A^B \sqrt{(g_{ab} + \delta g_{ab})\,d(x^a + \delta x^a)\,d(x^b + \delta x^b)} \tag{16}$$

Repeating the construction above, the integrand will contain terms of the form

$$\delta g_{ab}\,dx^a\,dx^b = g_{ab,c}\,\delta x^c\,dx^a\,dx^b \tag{17}$$

where we have introduced the notation of writing partial derivatives with a comma, so that

$$g_{ab,c} = \frac{\partial g_{ab}}{\partial x^c} \tag{18}$$

There will also be terms of the form

$$2g_{ab}\, dx^a\, d\delta x^b = 2g_{ac}\, dx^a\, d\delta x^c \tag{19}$$

where the factor of 2 is due to the symmetry between a and b, which in turn reflects the symmetric nature of the line element.

Expanding the square root, then dividing through by ds and integrating by parts as before leads to the general *geodesic equation*:

$$\left(g_{kc}\, \dot{x}^k\right)^{\cdot} = \frac{1}{2}\, g_{ab,c}\, \dot{x}^a\, \dot{x}^b \tag{20}$$

2.2   Example: The Sphere

The line element on the sphere is

$$ds^2 = r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{21}$$

We therefore have

$$g_{\theta\theta} = r^2 \qquad g_{\phi\phi} = r^2 \sin^2\theta \tag{22}$$

with all other components zero. The geodesic equation therefore becomes

$$
\begin{aligned}
r^2\, \ddot{\theta} &= \sin\theta \cos\theta\, \dot{\phi}^2 \\
r^2 \sin^2\theta\, \dot{\phi} &= \text{constant}
\end{aligned}
\tag{23}
$$

We can easily check these equations in two special cases, namely for the equator $(\theta = \frac{\pi}{2})$ and for meridians $(\phi = \text{constant})$.

3   Conserved Quantities

In both examples above, one of the geodesic equations implied that some quantity was constant. These particular *conserved quantities* are in fact the same: the angular momentum of an object in circular motion.

More generally, suppose that the metric coefficients g_{ab} are independent of one of the coordinates, x^c. Then the right-hand-side of the geodesic equation (20) vanishes, which implies that

$$g_{kc}\,\dot{x}^k = \text{constant} \tag{24}$$

Let v be the vector field whose components (in a coordinate basis) are

$$v^a = \delta^a{}_c \tag{25}$$

for some fixed value of c, so that

$$\nabla_v f = v \cdot \nabla f = \frac{\partial f}{\partial x^c} \tag{26}$$

and denote the tangent vector (velocity) to the given path by u, so that

$$u^a = \dot{x}^a \tag{27}$$

If $g_{ab,c} = 0$ ($\forall a, b$), then v is called a *Killing vector*, and the conserved quantity can be written as

$$u \cdot v = \text{constant} \tag{28}$$