4 Relativistic Mechanics

In which it is shown that mass is energy.

4.1 Proper Time

In the rest frame, position doesn't change. Let τ denote "wristwatch time" [2], that is, time as measured by a clock carried by an observer moving at constant speed u with respect to the given frame. In the moving observer's rest frame, position doesn't change. We therefore have

$$(\Delta x)^2 - c^2(\Delta t)^2 = 0 - c^2(\Delta \tau)^2 \tag{74}$$

so that

$$(\Delta \tau)^2 = \left(1 - \frac{1}{c^2}\left(\frac{\Delta x}{\Delta t}\right)^2\right)(\Delta t)^2 \tag{75}$$

or equivalently

$$d\tau = \sqrt{1 - \frac{u}{c^2}}\, dt = \frac{1}{\gamma}\, dt = \frac{1}{\cosh \alpha}\, dt \tag{76}$$

Note that proper time is independent of reference frame!

4.2 Energy and Momentum

Consider the *ordinary velocity* of a moving object, defined by

$$u = \frac{d}{dt}\, x \tag{77}$$

This transforms in a complicated way, since

$$\frac{1}{c}\frac{dx'}{dt'} = \frac{\frac{1}{c}\frac{dx}{dt} - \frac{v}{c}}{1 - \frac{v}{c^2}\frac{dx}{dt}} \tag{78}$$

The reason for this is that both the numerator and the denominator need to be transformed. The invariance of proper time suggests that we should instead differentiate with respect to proper time, since of course

$$\frac{d}{d\tau}\, x' = \frac{dx}{d\tau} \tag{79}$$

or in other words since the operator $\frac{d}{d\tau}$ pulls through the Lorentz transformation, so that only the numerator is transformed when changing reference frames.

Furthermore, the same argument can be applied to t, which suggests that there are (in 2 dimensions) 2 components to the velocity. We therefore consider the "2-velocity"

$$\boldsymbol{u} = \frac{d}{d\tau}\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} c\frac{dt}{d\tau} \\ \frac{dx}{d\tau} \end{pmatrix} \tag{80}$$

But since

$$dt = \cosh\alpha\, d\tau \tag{81}$$

and

$$dx^2 - c^2 dt^2 = -c^2 d\tau^2 \tag{82}$$

we also have

$$dx = c\,\sinh\alpha\, d\tau \tag{83}$$

so that

$$\boldsymbol{u} = c\begin{pmatrix} \cosh\alpha \\ \sinh\alpha \end{pmatrix} \tag{84}$$

Note that $\frac{1}{c}\,\boldsymbol{u}$ is a *unit* vector, that is

$$\frac{1}{c^2}\,\boldsymbol{u}\cdot\boldsymbol{u} = 1 \tag{85}$$

and further that

$$\frac{u}{c} = \frac{dx}{dt} = \tanh\alpha \tag{86}$$

as expected.

4.3 Conservation Laws

Suppose that (Newtonian) momentum is conserved in a given frame, that is

$$\sum m_i v_i = \sum \hat{m}_j \hat{v}_j \tag{87}$$

(Both of these would be 0 in the center-of-mass frame.) Changing to another frame moving with respect to the first at speed v, we have

$$v_i = v_i' + v \tag{88}$$

$$\hat{v}_j = \hat{v}_j' + v \tag{89}$$

so that

$$\sum m_i(v_i' + v) = \sum \hat{m}_j(\hat{v}_j' + v) \tag{90}$$

We therefore see that

$$\sum m_i v_i' = \sum \hat{m}_j \hat{v}_j' \iff \sum m_i = \sum \hat{m}_j \tag{91}$$

that is, momentum is conserved in *all* inertial frames provided it is conserved on one frame *and* mass is conserved.

Repeating the computation for the kinetic energy, we obtain starting from

$$\frac{1}{2}\sum m_i v_i^2 = \frac{1}{2}\sum \hat{m}_j \hat{v}_j^2 \tag{92}$$

that

$$\frac{1}{2}\sum m_i(v_i' + v)^2 = \frac{1}{2}\sum \hat{m}_j(\hat{v}_j' + v)^2 \tag{93}$$

Expanding this out, we discover that (kinetic) energy is conserved in all frames provided it is conserved in one frame *and* both mass and momentum are conserved.

The situation in special relativity is quite different.

Consider first the momentum defined by the ordinary velocity, namely

$$p = mu = m\frac{dx}{dt} \tag{94}$$

This momentum is *not* conserved!

We use instead the momentum defined by the 4-velocity, which is given by

$$p = m\frac{dx}{d\tau} = mc\sinh\alpha \tag{95}$$

Suppose now that, as seen in a particular inertial frame, the total momentum of a collection of particles is the same before and after some interaction, that is

$$\sum m_i c \sinh\alpha_i = \sum \hat{m}_j c \sinh\hat{\alpha}_j \tag{96}$$

Consider now the same situation as seen by another inertial reference frame, moving with respect to the first with speed

$$v = c\,\tanh\beta \tag{97}$$

We therefore have

$$\begin{aligned}\alpha_i &= \alpha_i' + \beta \qquad (98)\\ \hat{\alpha}_j &= \hat{\alpha}_j' + \beta \qquad (99)\end{aligned}$$

Inserting this into the conservation rule (96) leads to

$$\begin{aligned}\sum m_i c \sinh \alpha_i' &= \sum m_i c \sinh(\alpha_i - \beta) \qquad (100)\\ &= \left(\sum m_i c \sinh \alpha_i\right) \cosh \beta - \left(\sum m_i c \cosh \alpha_i\right) \sinh \beta \qquad (101)\end{aligned}$$

and similarly

$$\sum \hat{m}_j c \sinh \hat{\alpha}_j' = \left(\sum \hat{m}_j c \sinh \hat{\alpha}_j\right) \cosh \beta - \left(\sum \hat{m}_j c \cosh \hat{\alpha}_j\right) \sinh \beta \qquad (102)$$

The coefficients of $\cosh \beta$ in these 2 expressions are equal due to the assumed conservation of momentum in the original frame. We therefore see that conservation of momentum will hold in the new frame if and only if we have in addition that the coefficients of $\sinh \beta$ agree, namely

$$\sum m_i c \cosh \alpha_i = \sum \hat{m}_j c \cosh \hat{\alpha}_j \qquad (103)$$

But what is this?

4.4 Energy

This mystery is resolved by recalling that momentum is mass times velocity, and that there is also a "t-component" to the velocity. In analogy with the 2-velocity, we therefore define the "2-momentum" to be

$$\boldsymbol{p} = m \begin{pmatrix} c\frac{dt}{d\tau} \\ \frac{dx}{d\tau} \end{pmatrix} = mc \begin{pmatrix} \cosh \alpha \\ \sinh \alpha \end{pmatrix} \qquad (104)$$

The second term is clearly the momentum, which we denote by p, but what is the first term? If the object is at rest, $\alpha = 0$, and the first term is therefore just mc. But Einstein's famous equation

$$E = mc^2 \qquad (105)$$

leads us to suspect that this is some sort of energy. In fact, mc^2 is called the *rest energy* or *rest mass*.

In general, we *define* the energy of an object moving at speed $u = c \tanh \alpha$ to be the first component of $\boldsymbol{p}$, that is we define

$$\begin{aligned} E &:= mc^2 \cosh \alpha && (106) \\ p &:= mc \sinh \alpha && (107) \end{aligned}$$

or equivalently

$$\boldsymbol{p} = \begin{pmatrix} \frac{1}{c}E \\ p \end{pmatrix} \tag{108}$$

Is this definition reasonable? Consider the case $\frac{u}{c} \ll 1$. Then

$$\begin{aligned} E &= mc^2 \cosh \alpha = mc^2 \gamma && (109) \\ &= \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} && (110) \\ &\approx mc^2 + \frac{1}{2} mu^2 + \frac{3}{8} m \frac{u^4}{c^2} + ... && (111) \end{aligned}$$

The first term is the rest energy, the next term is the Newtonian kinetic energy, and the remaining terms are relativistic corrections to the kinetic energy.

The moral is that conservation of 2-momentum is equivalent to both conservation of momentum and conservation of energy, but that there is no requirement that the total mass be conserved.

Taking the (squared) norm of the 2-momentum, we obtain

$$-c^2 \boldsymbol{p} \cdot \boldsymbol{p} = E^2 - p^2 c^2 = m^2 c^4 \tag{112}$$

Note that this equation continues to makes sense if $m = 0$, although the expressions for E and p separately in terms of α or γ do not. In fact, γ must approach ∞, or equivalently $\frac{u^2}{c^2} = 1$, so that $|u| = c$; such particles *always* move at the speed of light!

We therefore conclude that there can be massless particles, which move at the speed of light, and which satisfy ($m = 0$ and)

$$E = |p|c \neq 0 \tag{113}$$

Photons are examples of such particles; quantum mechanically, one has $E = \hbar\nu$, where ν is the frequency of the light (and $\hbar = \frac{h}{2\pi}$ where h is Planck's constant.)

4.5 Useful Formulas

The key formulas for analyzing the collision of relativistic articles can all be derived from (106) and (107). Taking the difference of squares leads to the key formula (112) relating energy, momentum, and (rest) mass, which holds also for massless particles. Rewriting (106) and (107) leads directly to

$$\gamma = \cosh\alpha = \frac{E}{mc^2} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \tag{114}$$

and

$$\sinh\alpha = \frac{p}{mc} = \frac{u}{c}\gamma \tag{115}$$

and dividing these formulas yields

$$\tanh\alpha = \frac{pc}{E} = \frac{u}{c} \tag{116}$$

Finally, another useful formula is

$$\frac{m^2c^4}{E^2} = 1 - \frac{u^2}{c^2} = \left(1+\frac{u}{c}\right)\left(1-\frac{u}{c}\right) \approx 2\left(1-\frac{u}{c}\right) \tag{117}$$

where the final approximation holds if $u \approx c$.