1. **KEPLER'S LAW**

Consider circular (r=constant), timelike geodesic orbits in the plane $\theta = \frac{\pi}{2}$ of the Reissner-Nordström spacetime with metric

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)$$

with

$$f(r) = 1 - \frac{2m}{r} + \frac{e^2}{r^2}$$

where $e < m$ are constants, and with $r > m + \sqrt{m^2 - e^2}$, so that $f(r) > 0$.

(a) Using the r-component of the geodesic equation, or otherwise, derive an expression for the angular velocity $\Omega := \frac{d\phi}{dt}$ in terms of e, m, r.

(b) Show that in the limit $e = 0$ (Schwarzschild) this reduces to Kepler's law, namely $m = \Omega^2 r^3$.

Unlike Newtonian theory, it is not true in relativity that circular orbits exist for all values of r. For example, it turns out that stable circular orbits only exist in the Schwarzschild spacetime for $r > 6m$. It is also worth noting that an (unstable) circular null orbit exists at $r = 3m$, so that the gravitational field is strong enough to keep a beam of light in orbit! A further discussion of this point can be found in §6.3 of Wald.