MTH 437 **HW #4** Spring 2004

1. **THE STANDARD MODELS**

The line element for closed ($k = 1$) Robertson-Walker cosmologies can be written as

$$ds^2 = -d\tau^2 + a(\tau)^2 \left(d\psi^2 + \sin^2\psi\, d\theta^2 + \sin^2\psi \sin^2\theta\, d\phi^2\right)$$

You do not need to verify the stated form of Einstein's equations in the problems below.

(a) Consider the dust-filled ($p = 0$), closed ($k = 1$) Robertson-Walker cosmology. In this case, Einstein's equation (with $\Lambda = 0$) reduces to:

$$\frac{3(a,_\tau{}^2 + 1)}{a^2} = 8\pi\rho \quad ; \quad \rho a^3 = \frac{3q}{8\pi} = \text{constant}$$

and a parametric solution of these equations is given by

$$a = \frac{1}{2}q(1 - \cos\eta) \quad ; \quad \tau = \frac{1}{2}q(\eta - \sin\eta)$$

Draw a rough graph of a vs. τ for $\eta \in [0, 2\pi]$.

(b) Consider the radiation-filled ($p = \rho/3$), closed ($k = 1$) Robertson-Walker cosmology. In this case, Einstein's equation (with $\Lambda = 0$) reduces to:

$$\frac{3(a,_\tau{}^2 + 1)}{a^2} = 8\pi\rho \quad ; \quad \rho a^4 = \frac{3b^2}{8\pi} = \text{constant}$$

and a parametric solution of these equations is given by

$$a = b\sin\eta \quad ; \quad \tau = b(1 - \cos\eta)$$

Draw a rough graph of a vs. τ for $\eta \in [0, \pi]$.

(c) In both cases show that η satisfies the equation $d\eta/d\tau = 1/a$.
(η is called "conformal" time because the line element can now be written

$$ds^2 = a^2 \left(-d\eta^2 + d\psi^2 + \sin^2\psi\, d\theta^2 + \sin^2\psi \sin^2\theta\, d\phi^2\right)$$

and multiplying the metric by a function is called a conformal transformation.)

(d) Consider a radial (θ = constant, ϕ = constant), (initially) outgoing (ψ increasing), future pointing (τ increasing), null curve in these two cosmologies (and in fact in **any** closed Robertson-Walker cosmology). Show that $d\psi/d\tau = 1/a$ along the curve.

(e) Is such a curve a geodesic? Why or why not?
This question can be answered without computation. If you must use the geodesic equation, please only compute the Christoffel symbols you actually need!

(f) Show that a beam of light (i.e. a null geodesic) emitted radially from $\psi = 0$ at $\tau = 0$ goes precisely once around the dust-filled universe (a) during its lifetime, and precisely half way around the radiation-filled universe (b).

Extra Credit
Does this mean that one could see the back of one's head in either or both of these models?