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CURVATURE AND ALL THAT

Everything you ever wanted to know about curvature — and then some...

1. INTRODUCTION

This is a *very* concise summary of the basic idea of curvature, intended for those studying general relativity. Along the way, we encounter covariant differentiation and affine connections in considerable generality, and then discusses how to compute curvature in 2 important special cases, namely using a coordinate basis and using an orthonormal basis. The latter method requires familiarity with differential forms; the former does not.

Please bear in mind when reading this document that it is not necessary to follow the details of each and every step. Rather, it is important to have a basic grasp of what is going on, and to be able to calculate curvature using any *one* method.

While the presentation is aimed at those who are familiar with differential forms, it should be possible to follow most of the calculations with only a basic knowledge of tensors. (A *very* quick summary of tensors is given as an appendix.) Most formulas are also given in basis-free language.

2. THEORY

a) Curvature and Torsion

Pick *any* basis $\left\{\sigma^i\right\}$ of 1-forms, *not* necessarily orthonormal. Choose *any* 1-forms ${\omega^i}_j$ to be the *connection 1-forms*. Then the *torsion 2-forms* Θ^k and the *curvature 2-forms* ${\Omega^i}_j$ are defined by the *Cartan structure equations*

$$\Theta^k = d\sigma^k + {\omega^k}_i \wedge \sigma^i$$
$${\Omega^i}_j = d{\omega^i}_j + {\omega^i}_m \wedge {\omega^m}_j$$

Expanding with respect to our basis, we can write

$${\omega^k}_i =: {\Gamma^k}_{ij}\, \sigma^j$$
$$\Theta^k =: \frac{1}{2} {T^k}_{ij}\, \sigma^i \wedge \sigma^j$$
$${\Omega^i}_j =: \frac{1}{2} {R^i}_{jkl}\, \sigma^k \wedge \sigma^l$$

The latter 2 expressions correspond to a $\binom{1}{2}$ tensor called the *torsion tensor*, whose components are ${T^k}_{ij}$, and a $\binom{1}{3}$ tensor called the *Riemann curvature tensor*, whose components are ${R^i}_{jkl}$. The "connection components" ${\Gamma^k}_{ij}$ are called *Christoffel symbols*, and are *not* the components of a tensor. In particular, they can all be 0 in one basis but not in another, which is not possible for tensor components.

Two contractions of the Riemann tensor are important in relativity. These are the *Ricci tensor*, whose components are defined by $R_{ij} = {R^m}_{imj}$, and the *Ricci scalar*, which is the "trace" of the Ricci tensor, defined by $R = g^{ij} R_{ij}$, where g^{ij} denotes the (components of the) inverse of the metric tensor, which is discussed further below.

b) Covariant Differentiation

We are looking for a derivative operator ∇_X which takes tensors to tensors of the same rank. It should have at least the following properties (we will add more later):

$$\begin{aligned}
\nabla_X(f) &= X(f) \\
\nabla_{X+hY} T &= \nabla_X T + h \nabla_Y T \\
\nabla_X(S+T) &= \nabla_X S + \nabla_X T \\
\nabla_X(S \otimes T) &= (\nabla_X S) \otimes T + S \otimes (\nabla_X T) \\
\nabla_X\Big(\alpha(Y)\Big) &= \Big(\nabla_X \alpha\Big)(Y) + \alpha\Big(\nabla_X(Y)\Big)
\end{aligned}$$

The first of these says that covariant differentiation should reduce to ordinary differentiation when applied to functions, the next two are linearity requirements, the fourth is the usual product rule, and the last says that a "contraction" [1] can be done either before or after taking the derivative.

As outlined below, it is fairly easy to see that covariant differentiation is completely determined by its action on a basis. So let $\{e_i\}$ denote the basis of vector fields which is dual to the given basis $\left\{\sigma^i\right\}$ of 1-forms. Given a choice of connection 1-forms, defining

$$\nabla_{e_j} e_i := \omega^k{}_i(e_j)\, e_k$$

leads to a unique covariant differentiation operator satisfying the above requirements. Conversely, any such operator determines connection 1-forms via this equation. Thus, choosing a derivative operator is completely equivalent to choosing connection 1-forms.

To see that knowing the derivative of vector fields is enough, consider a 1-form α. The above requirements determine the derivative of α by computing

$$X\Big(\alpha(Y)\Big) = \nabla_X\Big(\alpha(Y)\Big) = \Big(\nabla_X \alpha\Big)(Y) + \alpha\Big(\nabla_X(Y)\Big)$$

for any vector fields X, Y. Inserting a basis 1-form for α, we obtain

$$\nabla_{e_j} \sigma^k = -\omega^k{}_i(e_j)\, \sigma^i$$

These formulas can be used to differentiate *any* tensor, by expanding with respect to an explicit basis and using the product rule. An explicit example is given by

$$\nabla_X Y = \Big(X(Y^k) + \omega^k{}_i(X) Y^i\Big)\, e_k$$

where $Y^k = \sigma^k(Y)$. Thus, the *components* of $\nabla_X Y$ are given by

$$X^j Y^k{}_{;j} := X^j \left(Y^k{}_{,j} + \Gamma^k{}_{ij} Y^i\right)$$

Note the conventional use of a comma to denote partial differentiation, that is $f_{,i} := e_i(f)$, and the corresponding use of a semicolon to denote covariant differentiation. Similarly, the components of $\nabla_X \alpha$ are

$$X^j \alpha_{i;j} := X^j \left(\alpha_{i,j} - \Gamma^k{}_{ij} \alpha_k\right)$$

In general, the derivative of a $\binom{p}{q}$ tensor will contain $p+q$ "correction terms" involving the connection $\omega^k{}_i$ (or equivalently $\Gamma^k{}_{ij}$), with appropriate signs.

[1] A contraction turns the $\binom{1}{1}$ tensor $\alpha \otimes X$ with components $\alpha_\mu X^\nu$ into the function $\alpha(X) = \alpha_\mu X^\mu$.

c) Example

Consider first Euclidean $\mathbb{R}^2$ in rectangular coordinates. Ordinary partial differentiation satisfies all the requirements, and can therefore be used for covariant differentiation as well. But the basis vector fields are constant, so that their derivatives vanish. Thus, *in this case*, covariant differentiation simply means to take the partial derivatives of the component functions. Equivalently, all the Christoffel symbols vanish; the connection 1-forms are identically 0.

Consider now the the same situation, but using the *coordinate* polar basis $\{dr, d\theta\}$, with dual basis[2] $\{e_r, e_\theta\} = \left\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right\}$. We wish to express the *same* covariant derivative in this basis. However, since the basis is not constant, we must take the derivatives of the basis as well as the components.

One way to do this is to express the polar basis in terms of the Cartesian basis $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$, which you can think of as $\{\vec{\imath}, \vec{\jmath}\}$. Since

$$\frac{\partial}{\partial r} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} \qquad\qquad \frac{\partial}{\partial \theta} = -r\sin\theta \frac{\partial}{\partial x} + r\cos\theta \frac{\partial}{\partial y}$$

direct computation yields

$$\nabla_{e_r} e_r = \frac{\partial e_r}{\partial r} = 0 \qquad \nabla_{e_r} e_\theta = \frac{\partial e_\theta}{\partial r} = \frac{1}{r} e_\theta = \frac{\partial e_r}{\partial \theta} = \nabla_{e_\theta} e_r \qquad \nabla_{e_\theta} e_\theta = \frac{\partial e_\theta}{\partial \theta} = -r\, e_r$$

Comparing this with the formula above we obtain

$$\omega^r{}_r = 0 \qquad \omega^\theta{}_\theta = \frac{1}{r}\, dr \qquad \omega^\theta{}_r = \frac{1}{r}\, d\theta \qquad \omega^r{}_\theta = -r\, d\theta$$

from which the Christoffel symbols can be read off.

Consider yet again the the same situation, this time using the *orthonormal* polar basis $\{dr, r\, d\theta\}$, with dual basis[2] $\left\{e_{\hat{r}}, e_{\hat{\theta}}\right\} = \left\{\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}\right\}$. This time we get

$$\frac{\partial}{\partial r} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} \qquad\qquad \frac{1}{r}\frac{\partial}{\partial \theta} = -\sin\theta \frac{\partial}{\partial x} + \cos\theta \frac{\partial}{\partial y}$$

and thus

$$\nabla_{e_{\hat{r}}} e_{\hat{r}} = \frac{\partial e_{\hat{r}}}{\partial r} = 0 = \frac{\partial e_{\hat{\theta}}}{\partial r} = \nabla_{e_{\hat{r}}} e_{\hat{\theta}} \qquad \nabla_{e_{\hat{\theta}}} e_{\hat{r}} = \frac{1}{r}\frac{\partial e_{\hat{r}}}{\partial \theta} = \frac{1}{r} e_{\hat{\theta}} \qquad \nabla_{e_{\hat{\theta}}} e_{\hat{\theta}} = \frac{1}{r}\frac{\partial e_{\hat{\theta}}}{\partial \theta} = -\frac{1}{r} e_{\hat{r}}$$

so that

$$\omega^{\hat{r}}{}_{\hat{r}} = 0 = \omega^{\hat{\theta}}{}_{\hat{\theta}} \qquad \omega^{\hat{\theta}}{}_{\hat{r}} = d\theta = -\omega^{\hat{r}}{}_{\hat{\theta}}$$

It is important to realize that the connection 1-forms in these 3 cases are quite different. In particular, the Christoffel symbols are identically 0 in rectangular coordinates, but not in the other — and therefore are *not* the components of a tensor. However, it is instructive to check by explicit computation that the torsion and curvature vanish for this connection, regardless of which basis is used to compute them; the torsion and curvature *are* tensors.

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[2] Note the conventional use of the coordinates, rather than numbers, to label the basis, and hats to distinguish the orthonormal case. Numerical indices may of course be used instead, but make sure you know which basis you're using!

d) The Torsion and Curvature, Revisited

In order to specify a preferred connection, we need expressions for the curvature and torsion. We outline here the lengthy calculation needed to derive these. You may wish to skip this section on first reading — or entirely.

Given a covariant derivative operator ∇, or equivalently given connection 1-forms $\omega^i{}_j$, the *torsion* tensor T is defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

and the *Riemann curvature* tensor R is defined by

$$R(X,Y)\,Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})\,Z$$

(It takes some work to show that these expressions in fact define tensors — the trick is to show that they are suitably multilinear.) In both cases, $[X,Y]$ denotes the *Lie bracket* or *commutator* of X and Y, which is the vector field defined by

$$[X,Y](f) = X\Big(Y(f)\Big) - Y\Big(X(f)\Big)$$

Using these definitions, one can work out the components of these tensors in a given basis in terms of the Christoffel symbols, obtaining (after some work!)

$$T^k{}_{ij}\, e_k := T(e_i, e_j) = -\left(\mathit{\Gamma}^k{}_{ij} - \mathit{\Gamma}^k{}_{ji}\right) e_k - [e_i, e_j]$$

and

$$R^i{}_{jkl} := \sigma^i\Big(R(e_k, e_l) e_j\Big) = e_k\left(\mathit{\Gamma}^i{}_{jl}\right) - e_l\left(\mathit{\Gamma}^i{}_{jk}\right) + \mathit{\Gamma}^i{}_{mk}\mathit{\Gamma}^m{}_{jl} - \mathit{\Gamma}^i{}_{ml}\mathit{\Gamma}^m{}_{jk} - \mathit{\Gamma}^i{}_{jm}[e_k, e_l]^m$$

Using these expressions, as well as the important identity

$$2\,d\alpha(X,Y) = X\Big(\alpha(Y)\Big) - Y\Big(\alpha(X)\Big) - \alpha\Big([X,Y]\Big)$$

for any 1-form α and vector fields X, Y, one can further verify (again after some work!) that the components $T^k{}_{ij}$ and $R^i{}_{jkl}$ agree with those defined above using differential forms! [3]

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[3] It is logically cleaner to proceed in this direction, rather than the other way around, since the earlier expressions must be *assumed* to be antisymmetric in their final 2 indices, albeit without loss of generality.

e) The Metric Connection

Again, this section outlines how to calculate an important result, the details of which can be skipped on first reading.

We now assume the existence of a metric tensor and impose 2 additional, desirable properties on the connection, namely that it be *torsion-free* and *metric compatible*, and show that there is a unique connection with these properties, called the *Levi-Civita* connection.

The vanishing of the torsion tells us that

$$[e_i, e_j] + \left(\Gamma^k{}_{ij} - \Gamma^k{}_{ji}\right) e_k = 0$$

or equivalently that

$$g([e_i, e_j], e_k) + \omega_{ki}(e_j) - \omega_{kj}(e_i) = 0$$

where we have introduced the notation $\omega_{kj} := g_{ki}\,\omega^k{}_j$, and where $g_{ki} = g(e_k, e_i)$ are the components of the metric tensor.

Metric-compatibility says that the covariant derivative of the metric tensor should vanish, that is $\nabla_X g = 0$. Using the contraction property in the definition of covariant differentiation, we have

$$\begin{aligned} X\Big(g(Y,Z)\Big) &= \nabla_X\Big(g(Y,Z)\Big) \\ &= (\nabla_X g)(Y,Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ &= 0 + g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \end{aligned}$$

Inserting basis vectors for X, Y, Z leads to

$$\begin{aligned} e_k(g_{lm}) &= g(\nabla_{e_k} e_l, e_m) + g(e_l, \nabla_{e_k} e_m) \\ &= g(\omega^n{}_l(e_k) e_n, e_m) + g(e_l, \omega^n{}_m(e_k) e_n) \\ &= g_{mn}\omega^n{}_l(e_k) + g_{ln}\omega^n{}_m(e_k) \\ &= \omega_{ml}(e_k) + \omega_{lm}(e_k) \end{aligned}$$

Taking a nonobvious combination of such terms, we obtain

$$\begin{aligned} e_k(g_{lm}) + e_m(g_{lk}) - e_l(g_{km}) &= \omega_{ml}(e_k) + \omega_{kl}(e_m) - \omega_{mk}(e_l) \\ &\qquad + \omega_{lm}(e_k) + \omega_{lk}(e_m) - \omega_{km}(e_l) \\ &= 2\,\omega_{lm}(e_k) - g(e_k, [e_l, e_m]) - g(e_m, [e_l, e_k]) - g(e_l, [e_m, e_k]) \end{aligned}$$

where the last step involves clever pairing of the terms and repeated use of the fact that the torsion vanishes. Solving for ω_{lm}, we obtain the *Koszul formula* for the Levi-Civita connection, namely

$$\begin{aligned} 2\,\omega_{lm}(e_k) &= e_k(g_{lm}) + e_m(g_{lk}) - e_l(g_{km}) \\ &\qquad + g(e_k, [e_l, e_m]) + g(e_m, [e_l, e_k]) - g(e_l, [e_k, e_m]) \end{aligned}$$

3. PRACTICE

a) Coordinate Basis

In a coordinate basis $\{dx^\mu\}$, several simplifications take place. First of all, the dual basis consists of "pure" partial derivative operators (with no coefficients), which means further that the commutator of any dual basis vector fields vanishes, since mixed partial derivatives can be taken in any order. In addition, since each basis 1-form is exact, its exterior derivative vanishes.

Starting with the connection, we have

$$\omega^\kappa{}_\mu = \Gamma^\kappa{}_{\mu\nu}\, dx^\nu$$

Note that $\Gamma^\kappa{}_{\mu\nu} = \Gamma^\kappa{}_{\nu\mu}$, since the torsion vanishes, and the Koszul formula yields

$$2\,\Gamma_{\kappa\mu\nu} = g_{\kappa\mu,\nu} + g_{\kappa\nu,\mu} - g_{\nu\mu,\kappa}$$

where $\Gamma_{\kappa\mu\nu} = g_{\kappa\lambda}\Gamma^\lambda{}_{\mu\nu}$ and where commas denote partial differentiation, namely $f_{,\mu} = \frac{\partial f}{\partial x^\mu}$. This is usually written in the form

$$\Gamma^\kappa{}_{\mu\nu} = \frac{1}{2} g^{\kappa\lambda}\left(g_{\kappa\mu,\nu} + g_{\kappa\nu,\mu} - g_{\nu\mu,\kappa}\right)$$

in terms of the *inverse metric* $g^{\kappa\lambda}$, which satisfies $g^{\kappa\lambda}g_{\lambda\mu} = \delta^\kappa{}_\mu$.

The curvature can be obtained either from the general formula above or by computing the curvature 2-form. In either case, one obtains

$$R^\mu{}_{\nu\kappa\lambda} = \Gamma^\mu{}_{\nu\lambda,\kappa} - \Gamma^\mu{}_{\nu\kappa,\lambda} + \Gamma^\mu{}_{\rho\kappa}\Gamma^\rho{}_{\nu\lambda} - \Gamma^\mu{}_{\rho\lambda}\Gamma^\rho{}_{\nu\kappa}$$

b) Orthonormal Basis

Now consider the case of an orthonormal basis $\{e_i\}$, that is, one which satisfies $g(e_i, e_j) = \pm\delta_{ij}$. Since the metric components are constant, their partial derivatives vanish. The Koszul formula yields for the connection

$$2\,\omega_{lm}(e_k) = g(e_k, [e_l, e_m]) + g(e_m, [e_l, e_k]) - g(e_l, [e_k, e_m])$$

However, this is not usually the simplest approach.

The Koszul formula guarantees us a *unique* torsion free, metric compatible connection. Thus, there is a *unique* solution of the first Cartan structure equation with vanishing torsion, which also satisfies the metric compatibility condition, which now reads

$$e_k(g_{lm}) = 0 = \omega_{ml}(e_k) + \omega_{lm}(e_k)$$

or simply $\omega_{ml} + \omega_{lm} = 0$. In practice, it is often easiest to solve the equations [4]

$$d\sigma^k + \omega^k{}_i \wedge \sigma^i = 0$$

subject to the condition

$$\omega_{ji} = -\omega_{ij}$$

Feel free to guess a solution — if it works, you're done! The curvature is then easily calculated from the remaining structure equation

$$R^i{}_{jkl}\, \sigma^k \wedge \sigma^l = d\omega^i{}_j + \omega^i{}_m \wedge \omega^m{}_j$$

[4] Note the position of the indices! Care must be taken in Lorentzian signature.

4. TENSOR-VALUED FORMS *(This section is optional! Compare §14.5 of MTW.)*

There is an elegant way to generalize the exterior derivative from differential forms to tensors of all types. This is done by first considering "tensor-valued" differential forms, whose "components" are tensors. A good example is the vector-valued 1-form [5]

$$d\mathcal{P} = e^i \otimes \sigma_i$$

which is really a $\binom{1}{1}$ tensor (the identity matrix!). Working by analogy with

$$df(X) = \nabla_X f$$

for functions f and vector fields Y, we require

$$dY(X) = \nabla_X Y$$

Comparing this with the formula on the first page, namely

$$\nabla_X Y = \left(X(Y^k) + \omega^k{}_i(X)Y^i\right) e_k = \left[\left(dY^k + Y^i\omega^k{}_i\right)(X)\right] e_k$$

we are led to require

$$de_i := e_k \otimes \omega^k{}_i$$

so that

$$d(Y^i e_i) = d(e_i Y^i) = e_i \otimes dY^i + de_i \otimes Y^i = e_k \otimes \left(dY^k + \omega^k{}_i Y^i\right)$$

It is now possible to take covariant derivatives of *contravariant* tensors (tensors of type $\binom{p}{0}$) simply by applying this generalized exterior derivative d! [6] Don't forget the product rule for d, which means in particular that

$$d(S \otimes \alpha) = dS \wedge \alpha + S \otimes d\alpha$$

for tensors S and forms α.

Two computations are particularly nice in this formalism. First of all,

$$\begin{aligned} d^2\mathcal{P} := d(d\mathcal{P}) &= de_i \wedge \sigma^i + e_k \otimes d\sigma^k \\ &= e_k \otimes \omega^k{}_i \wedge \sigma^i + e_k \otimes \left(\Theta^k - \omega^k{}_i \wedge \sigma^i\right) = e_k \otimes \Theta^k \end{aligned}$$

In particular, the condition that the torsion vanish is just the statement that $d^2\mathcal{P} = 0$. Similarly, considering an arbitrary vector field Y, we have

$$\begin{aligned} d^2Y = d(dY) &= d\left(e_k \otimes \left(dY^k + \omega^k{}_j Y^j\right)\right) \\ &= de_k \wedge \left(dY^k + \omega^k{}_j Y^j\right) + e_i \otimes d\left(dY^i + Y^j\omega^i{}_j\right) \\ &= e_i \otimes \omega^i{}_k \wedge \left(dY^j + Y^j\omega^k{}_j\right) + e_i \otimes \left(0 + dY^j \wedge \omega^i{}_j + Y^j d\omega^i{}_j\right) \\ &= e_i \otimes \left(d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j\right) Y^j = e_i \otimes \Omega^i{}_j Y^j \end{aligned}$$

so that d^2 acting on vectors gives the curvature.

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[5] The "d" in $d\mathcal{P}$ should be thought of as part of the name, although it can be motivated by considering $d\vec{\boldsymbol{r}} = \vec{\boldsymbol{\imath}}\,dx + \vec{\boldsymbol{\jmath}}\,dy$ in rectangular coordinates in Euclidean $\mathbb{R}^2$.

[6] While this can be extended to all tensors, some confusion results, since the derivative of the *tensor* σ^i differs from the derivative of the *1-form* σ^i.

APPENDIX: Tensors

We start with an arbitrary surface M ("smooth manifold"), described in terms of suitable coordinates x^μ. There are usually many admissible coordinates, such as polar and rectangular coordinates on (most of) $\mathbb{R}^2$.

A *vector field* X on M, often simply called a *vector*, is a differential operator which acts on functions on M by taking their directional derivative in the direction X. For instance, $\frac{\partial}{\partial x^\mu}$ is a vector field, and any vector field can be written as $X = X^\mu \frac{\partial}{\partial x^\mu}$, where the components X^μ are functions on M, and where we have introduced the *Einstein summation convention* that repeated indices are summed. The vector fields $\left\{\frac{\partial}{\partial x^\mu}\right\}$ thus form a basis, called a *coordinate basis*. You can think of this as replacing the usual Euclidean basis vector $\vec{\imath}$ by the differential operator $\frac{\partial}{\partial x}$, and so forth. There is of course a coordinate basis for each set of coordinates; the different bases are related by Jacobian transformations.

A *1-form* α on M is a linear map which acts on vector fields. We start with the *differential* df of a function f on M, and define $df(X) := X(f)$; both of these expressions yield the directional derivative of f in the direction of X. We therefore have $dx^\mu(\frac{\partial}{\partial x^\nu}) = \delta^\mu{}_\nu$, where δ denotes the Kronecker delta. The basis of coordinate differentials $\{dx^\mu\}$ is therefore dual to the above coordinate basis of vector fields, and in particular $dx^\mu(X) = X^\mu$. An arbitrary 1-form α can be written as $\alpha = \alpha_\mu dx^\mu$ in this basis, where the components α_μ are again functions on M. Finally, $\alpha(X) = \alpha_\mu X^\mu$.

A *tensor* of type $\binom{p}{q}$ is a multilinear map which acts on p 1-forms and q vector fields. A 1-form is thus a tensor of type $\binom{0}{1}$ and, since the dual of the dual of a vector space is the original vector space, a vector field is a tensor of type $\binom{1}{0}$. Functions are regarded as $\binom{0}{0}$ tensors. A very important tensor is the *metric tensor* g (if there is one!), which has type $\binom{0}{2}$. This means it acts on 2 vector fields — giving their "dot product", that is $g(X,Y) = X \cdot Y$. A basis for tensors of a given type can be constructed from the bases for vectors and 1-forms. For instance, the metric can be expanded as $g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$, where the tensor product $\otimes$ means that $(dx^\mu \otimes dx^\nu)\,(X,Y) = dx^\mu(X)\, dx^\nu(Y)$. The metric for (Euclidean) $\mathbb{R}^2$ in rectangular coordinates is $g = dx \otimes dx + dy \otimes dy$, which should look familiar!