Integrals

$\vec{e} = \frac{d\vec{\alpha}}{du}$

$\Rightarrow du(\vec{e}) = \vec{e}[u] = 1$

$\Rightarrow \vec{F}\cdot d\vec{\alpha} = \vec{F}\cdot\vec{e}\, du =: \phi$ ← 1-form!

$\Rightarrow \underline{\phi(\vec{e}) = \vec{F}\cdot\vec{e}}$

$$\therefore \int_\alpha \phi = \int_I \alpha^*\phi = \int_a^b \phi(\vec{e})\, du$$

$$= \int_a^b \vec{F}\cdot d\vec{\alpha} \quad \text{\underline{work!}}$$

$\vec{e}_1 = \frac{\partial\vec{x}}{\partial u}$, $\vec{e}_2 = \frac{\partial\vec{x}}{\partial v}$

$\Rightarrow du\wedge dv(\vec{e}_1, \vec{e}_2) = 1$

Now, let $d\vec{x}_1 = \frac{\partial\vec{x}}{\partial u}du = \vec{e}_1 du$ & $d\vec{x}_2 = \frac{\partial\vec{x}}{\partial v}dv = \vec{e}_2 dv$

$\Rightarrow \vec{F}\cdot d\vec{x}_1\times d\vec{x}_2 = \vec{F}\cdot\vec{e}_1\times\vec{e}_2\, du\wedge dv =: \eta$ ← 2-form!

$\Rightarrow \underline{\eta(\vec{e}_1, \vec{e}_2) = \vec{F}\cdot\vec{e}_1\times\vec{e}_2}$

$$\Rightarrow \int_x \eta = \int_R x^*\eta = \int_a^b\int_c^d \eta(\vec{e}_1, \vec{e}_2)\, du\, dv$$

$$= \int \vec{F}\cdot d\vec{A} \quad \text{\underline{flux!}}$$

Notation

$$d\vec{A} = d\vec{x}_1\times d\vec{x}_2 = \frac{\partial\vec{x}}{\partial u}du\times\frac{\partial\vec{x}}{\partial v}dv = \vec{e}_1\times\vec{e}_2\, du\, dv$$

$$\therefore \vec{F}\cdot d\vec{r} \leftrightarrow \vec{F}\cdot\vec{e}\, du$$

$$\vec{F}\cdot d\vec{A} \leftrightarrow \vec{F}\cdot\vec{e}_1\times\vec{e}_2\, du\wedge dv$$

∃2 types of integrals:

<u>measure theory</u> — $\int f\,ds$, $\int f\,dA$, $\int f\,dV$

<u>positive</u> — $\uparrow |d\vec{\alpha}|$, $\uparrow |d\vec{x}_1\times d\vec{x}_2|$

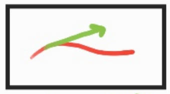
<u>differential forms</u> — $\int f\,du$, $\int f\,du\wedge dv$, $\int f\,du\wedge dv\wedge dw$

<u>oriented!</u> — $\uparrow \int \vec{F}\cdot d\vec{r}$, $\uparrow \int \vec{F}\cdot d\vec{x}_1\times d\vec{x}_2$

Vector Fields

$\vec{F} \in TM$

$\vec{\alpha} : I \to M$



$\vec{e} = \frac{d\vec{\alpha}}{du}$

Work: $\int_C \vec{F} \cdot d\vec{r} \mapsto \int_\alpha \vec{F} \cdot d\vec{\alpha} = \int_a^b \vec{F} \cdot \vec{e} \, du$

$F = \vec{F} \cdot d\vec{\alpha}$ is a 1-form!

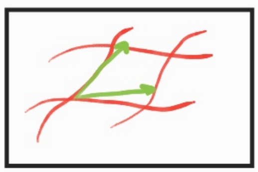
$F(\vec{e}) = \vec{F} \cdot \vec{e} \, du(\vec{e}) = \vec{F} \cdot \vec{e}$

$\therefore$ F takes $\vec{v}$ to $\vec{F} \cdot \vec{v}$ for $\vec{v}$ tangent to α

(true for any α)

" $F = \vec{F} \cdot$ "

$\vec{x} : R \to M$



$\vec{e}_1 = \frac{\partial \vec{x}}{\partial u}$

$\vec{e}_2 = \frac{\partial \vec{x}}{\partial v}$

Flux: $\int_S \vec{F} \cdot d\vec{A} \mapsto \int_R \vec{F} \cdot d\vec{x}_1 \times d\vec{x}_2 = \int_a^b \int_c^d \vec{F} \cdot \vec{e}_1 \times \vec{e}_2 \, du \, dv$

$Q = \vec{F} \cdot d\vec{x}_1 \times d\vec{x}_2$ is a 2-form!

$$\begin{aligned} Q(\vec{e}_1, \vec{e}_2) &= \vec{F} \cdot \vec{e}_1 du \times \vec{e}_2 dv \ (\vec{e}_1, \vec{e}_2) \\ &= \vec{F} \cdot \vec{e}_1 \times \vec{e}_2 \, du(\vec{e}_1) dv(\vec{e}_2) \\ &= \vec{F} \cdot \vec{e}_1 \times \vec{e}_2 \end{aligned}$$

$\therefore$ can identify Q with $\vec{F} \cdot \vec{e}_1 \times \vec{e}_2 \, du \wedge dv$

Forms

$$\vec{e}_1 = \frac{\partial x_i}{\partial u}\hat{x}_i \ , \ \vec{e}_2 = \frac{\partial x_i}{\partial v}\hat{x}_i$$

$$\Rightarrow \vec{e}_1 \times \vec{e}_2 = \frac{1}{2}\frac{\partial(x_i,x_j)}{\partial(u,v)}\hat{x}_i \times \hat{x}_j = \frac{1}{2}\frac{\partial(x_i,x_j)}{\partial(u,v)}\epsilon_{ijk}\hat{x}_k$$

$$\vec{F} = F_k\hat{x}_k \Rightarrow \vec{F}\cdot\vec{e}_1\times\vec{e}_2 = \frac{1}{2}\frac{\partial(x_i,x_j)}{\partial(u,v)}\epsilon_{ijk}F_k$$

But $\vec{x} = x_i\hat{x}_i \Rightarrow d\vec{x} = dx_i\hat{x}_i$

$$\Rightarrow F = \vec{F}\cdot d\vec{r} = F_k dx_k$$

$$\Rightarrow *F = \frac{1}{2}F_k\epsilon_{ijk}\,dx_i\wedge dx_j$$
$$= \frac{1}{2}F_k\epsilon_{ijk}\frac{\partial(x_i,x_j)}{\partial(u,v)}du\wedge dv$$

$$\therefore Q = \vec{F}\cdot d\vec{x}_1\times d\vec{x}_2$$
$$= F\cdot\vec{e}_1\times\vec{e}_2\,du\wedge dv$$
$$= *F$$

$\therefore$ <u>work</u>: $\int_c \vec{F}\cdot d\vec{r} \mapsto \int_\alpha F$

<u>flux</u>: $\int_S \vec{F}\cdot d\vec{A} \mapsto \int_x *F$