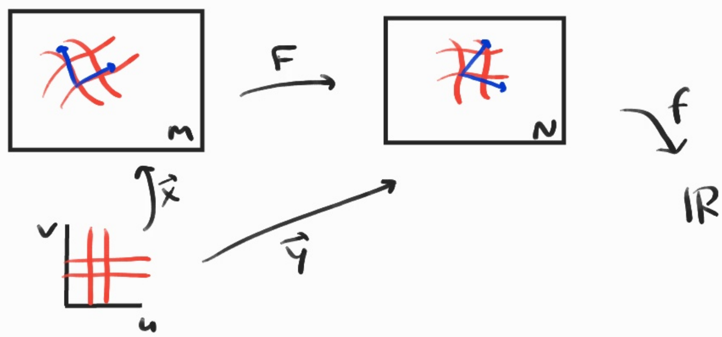
Maps

special case: $M = \mathbb{R}^2 \subset \mathbb{R}^3$!



$$\vec{y} = F \circ \vec{x}$$

$$F_* : TM \longrightarrow TN$$ "push forward"

$$\frac{\partial \vec{x}}{\partial u} \longmapsto \frac{\partial \vec{y}}{\partial u}$$

If f, Φ, η are forms on $\underline{\underline{N}}$

Then:

$$\boxed{\begin{aligned} F^* f &= f \circ F \\ F^*\Phi(\vec{v}) &= \Phi(F_* \vec{v}) \\ F^*\eta(\vec{v},\vec{w}) &= \eta(F_*\vec{v}, F_*\vec{w}) \end{aligned}}$$ "pullback"

__Thm__: F^* commutes with $+, \wedge, d$

$\therefore$ in practice, "use what you know"

__Ex__: $M = \{g = \text{const}\}$

$\Rightarrow dg = 0$

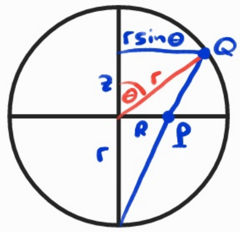
__Ex__: $x = r\cos\theta$

$\Rightarrow dx = dr\cos\theta - r\sin\theta\, d\theta$

cylinder? $r = \text{const} \Rightarrow dr = 0$

$\Rightarrow dx = -r\sin\theta\, d\theta$

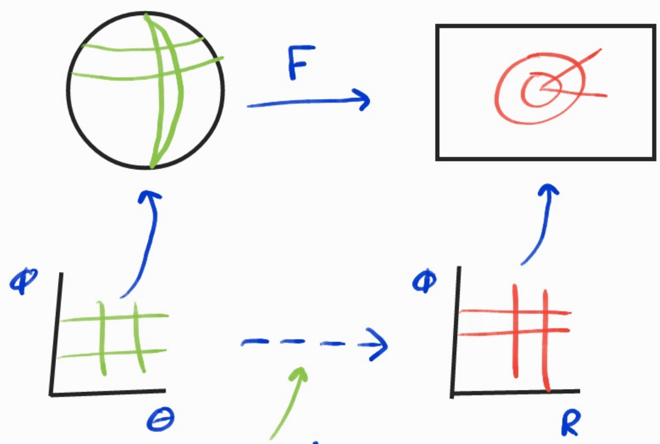
Example: (Stereographic Projection)



$$P = (R\cos\Phi,\ R\sin\Phi,\ 0)$$
$$Q = (r\sin\theta\cos\Phi,\ r\sin\theta\sin\Phi,\ r\cos\theta)$$

$M = S^2$ $\qquad (\theta, \Phi) \mapsto Q$

$N =$ xy-plane $\qquad (R, \Phi) \mapsto P$



In practice, just need relationship between R & θ:

$$\frac{R}{r} = \frac{r\sin\theta}{r + r\cos\theta}$$

$$\Rightarrow R = \frac{r\sin\theta}{1+\cos\theta} = \frac{2r\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cancel{1}+(2\cos^2\frac{\theta}{2}-\cancel{1})} = \underline{r\tan\frac{\theta}{2}}$$

$$\Rightarrow \boxed{dR = \frac{r\,d\theta}{1+\cos\theta}}$$

Line Integrals

Idea: Just do it!

Example: Integrate $x\,dy$ around circle

$$\Rightarrow x = r\cos\theta,\ y = r\sin\theta$$

$$\Rightarrow dy = r\cos\theta\, d\theta$$

$$\Rightarrow \oint_c x\,dy = \int_0^{2\pi} r^2\cos^2\theta\, d\theta = \pi r^2$$

Theory:

$$\alpha : [0, 2\pi] \longrightarrow \mathbb{R}^2$$

$$\theta \longmapsto (r\cos\theta, r\sin\theta)$$

Basis of $T\mathbb{R}$ is $\vec{e}$:

$$\vec{e}[f(\theta)] = \frac{df}{d\theta}$$

$$\Rightarrow \alpha_*(\vec{e}) = \frac{d\vec{\alpha}}{d\theta} = -r\sin\theta\,\hat{x} + r\cos\theta\,\hat{y} = r\hat{\theta}$$

$$= -y\hat{x} + x\hat{y}$$

$$\Rightarrow \frac{d\vec{\alpha}}{d\theta}[g(x,y)] = \alpha_*\vec{e}[g]$$

$$= \vec{e}[\alpha^* g] = \vec{e}[g\circ\alpha]$$

$$= \frac{d}{d\theta}\left(g(\alpha(\theta))\right) = \frac{\partial g}{\partial x}\frac{dx}{d\theta} + \frac{\partial g}{\partial y}\frac{dy}{d\theta}$$

$$= -\frac{\partial g}{\partial x} r\sin\theta + \frac{\partial g}{\partial y} r\cos\theta$$

$$\& \Rightarrow \alpha^*(x\,dy)(\vec{e}) = x\,dy\,(\alpha_*\vec{e})$$

$$= x\,dy\,(r\hat{\theta}) = x^2 \ \underline{\underline{\text{on } \alpha}}$$

$$= r^2\cos^2\theta$$

Def: $$\boxed{\int_\alpha \Phi = \int_I \alpha^*\Phi = \int_a^b \Phi\left(\frac{d\alpha}{du}\right) du}$$

where $\alpha : I \to M$