Connection

$$d\sigma_1 = \omega_{12} \wedge \sigma_2$$
$$\Rightarrow d\sigma_1(\hat{e}_1, \hat{e}_2) = \omega_{12}(\hat{e}_1)$$
$$d\sigma_2 = -\omega_{12} \wedge \sigma_1$$
$$= \sigma_1 \wedge \omega_{12}$$
$$\Rightarrow d\sigma_2(\hat{e}_1, \hat{e}_2) = \omega_{12}(\hat{e}_2)$$
$$\therefore \boxed{\omega_{12} = d\sigma_1(\hat{e}_1, \hat{e}_2)\sigma_1 + d\sigma_2(\hat{e}_1, \hat{e}_2)\sigma_2}$$

check: sphere: $\sigma_1 = r\,d\theta, \quad \sigma_2 = r\sin\theta\, d\phi$
$$\Rightarrow d\sigma_1 = 0, \quad d\sigma_2 = r\cos\theta\, d\theta \wedge d\phi$$
$$= \frac{\cot\theta}{r}\sigma_1 \wedge \sigma_2$$
$$\Rightarrow \omega_{12} = 0 + \frac{\cot\theta}{r}\sigma_2 = \cos\theta\, d\phi$$

Isometry

$$\sigma_i = F^*(\tau_i)$$
$$\omega_{ij} = F^*(\chi_{ij})$$

$$\begin{tikzcd} M \arrow[r, "F"] & N \\ \hat{e}_i \arrow[r, mapsto] \arrow[d, dash] & \hat{f}_i \arrow[d, dash] \\ \sigma_i \arrow[d, dash] & \tau_i \arrow[d, dash] \\ \omega_{ij} & \chi_{ij} \end{tikzcd}$$

$$\therefore F^* K = K\,!$$

$$\boxed{\text{Gaussian curvature is isometric invariant}}$$