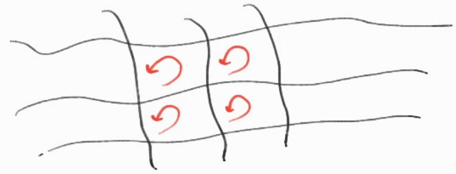
# Gauss-Bonnet Theorem

Idea: consider rectangular paving
of compact surface



$$\oint_{\partial R} k_g\,ds + \int_R K\,dM + \sum(\pi - \eta_i) = 2\pi$$

$\therefore$ 1 rectangle: $\oint k_g\,ds + \int K\,dM = \sum \eta_i - 2\pi$

now sum over rectangles:

- $\sum \oint k_g\,ds = 0$ due to opposite orientations
- $\sum\sum \eta_i = 2\pi v$
- $\sum 2\pi = 2\pi f$
- 4 edges per face, each in 2 faces
  $\Rightarrow e = 2f \Rightarrow f = e - f$

$$\therefore \int_M K\,dM = 2\pi(v - f) = 2\pi(v - e + f) = 2\pi\chi(M)$$

$$\boxed{\int_M K\,dM = 2\pi\chi(M)}$$

geometry　　topology

## Nonsingular vector fields

Hard to find! (on compact surface)

Ex: $\frac{\partial \vec{x}}{\partial \psi}, \frac{\partial \vec{x}}{\partial \phi}$ on torus

Thm: $M$ compact, $\vec{v} \in TM$, $\vec{v} \neq 0$ anywhere
$\Rightarrow M = $ torus

Pf: $\vec{v} \neq 0 \mapsto$ choose $\hat{e}_1 = \frac{\vec{v}}{|\vec{v}|}$, $\hat{e}_2 = J(\hat{e}_1)$ everywhere

$$\Rightarrow 2\pi \chi = \int_M K\,dM = -\int_M d\omega_{12} = -\underbrace{\int_{\partial M}}_{``\emptyset"} \omega_{12} = 0$$

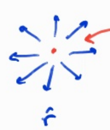
"can not comb hair on sphere"

---

## Index

Idea: compare angle of vector field with frame field

Ex:




$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$
$\Rightarrow$ angle to $\hat{x}$ is $\theta$
$\Rightarrow \oint_C d\theta = 2\pi \cdot 1$

$$\boxed{\mathrm{ind}(\hat{r}, 0) = 1}$$

"index of $\hat{r}$ at origin"

In general, "near" singularity, use any frame:


$$\int_D K\,dM = -\int_D d\omega_{12} = -\oint_C \omega_{12}$$

& away from singularity use $\hat{e}_1 = \frac{\vec{v}}{|\vec{v}|}$ as above:

$$\int_{M-D} K\,dM = -\int_{M-D} d\bar{\omega}_{12} = -\int_{\partial(M-D)} \bar{\omega}_{12} = +\int_{\partial D} \bar{\omega}_{12} = +\oint_C \bar{\omega}_{12}$$

Now let $\vec{q}$ be parallel on $C$

$$\Rightarrow \underset{\text{angle to frame}}{d\varphi} + \omega_{12} = 0 = \underset{\text{angle to } \vec{v}}{d\bar{\varphi}} + \bar{\omega}_{12}$$

$$\Rightarrow \int_M K\,dM = \oint_C \bar{\omega}_{12} - \omega_{12} = \oint_C d(\underbrace{\varphi - \bar{\varphi}}_{\text{angle from } \vec{v} \text{ to frame!}}) = 2\pi\,\mathrm{ind}(\vec{v}, p)$$

$$\therefore \boxed{\chi(M) = 2\pi \underbrace{\sum}_{\text{sum over all singularities}} \mathrm{ind}(\vec{v}, p)}$$

Poincaré – Hopf