Functions

$T(x,y) = k(x^2+y^2)$

$\Rightarrow T(r,\theta) = ?$

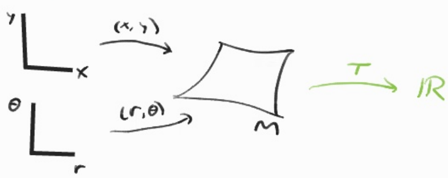
ⓐ kr^2

ⓑ $k(r^2+\theta^2)$

① $T = f(x,y) = k(x^2+y^2)$
$= g(r,\theta) = kr^2$

② $T(x,y) = k(x^2+y^2)$
$T(r,\theta) = kr^2$

③



$\text{"}T(x,y)\text{"} = T \circ (x,y)$

$\text{"}T(r,\theta)\text{"} = T \circ (r,\theta)$

Science is about relationships between physical quantities

$T = k(x^2+y^2) = kr^2$

equations, not functions

scalar fields!

Properties of Geodesics

Lemma: $\vec{v}$ geodesic $\Rightarrow |\vec{v}| = \text{const}$

Pf: $\nabla_{\vec{v}} \langle \vec{v}, \vec{v} \rangle = 2 \nabla_{\vec{v}} \vec{v} \cdot \vec{v} = 0$

Converse almost true:
$|\vec{v}| = \text{const}$ is a 1^{st}-order ODE
that usually eliminates one 2^{nd}-order ODE

Example: polar coords

2^{nd}-order:
$$\ddot{r} - r\dot{\theta}^2 = 0$$
$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} = 0$$

$$\vec{v} = \dot{r}\vec{e}_1 + \dot{\theta}\vec{e}_2$$
$$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$
$$\Rightarrow |\vec{v}|^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

$$\left[\left(\frac{ds}{du}\right)^2 = \frac{dr^2 + r^2 d\theta^2}{du^2}\right]$$

$\therefore |\vec{v}| = \text{const}$
$$\Rightarrow 2\dot{r}\ddot{r} + 2r\dot{r}\dot{\theta}^2 + 2r^2\dot{\theta}\ddot{\theta} = 0$$
$$\Rightarrow 2\dot{r}(\ddot{r} - r\dot{\theta}^2) + 2r^2\dot{\theta}\left(\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta}\right)$$

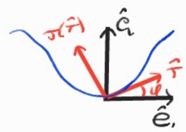
$\therefore$ eq1 $= 0 \Leftrightarrow$ eq2 $= 0$
so long as $\dot{r}\dot{\theta} \neq 0$
(and $|\vec{v}| = \text{const}$)

Geodesic Curvature

Recall Darboux frame

$$\frac{d\hat{T}}{dt} = g\,\underbrace{\hat{n}\times\hat{T}}_{J(\hat{T})} + k\,\hat{n}$$

Here



$$\hat{T} = \cos\varphi\,\hat{e}_1 + \sin\varphi\,\hat{e}_2$$

$$\Rightarrow \nabla_{\hat{T}}\hat{T} = (-\sin\varphi\,\hat{e}_1 + \cos\varphi\,\hat{e}_2)\,\underbrace{\hat{T}[\varphi]}_{\frac{d\varphi}{ds}} + \cos\varphi\,\omega_{12}(\hat{T})\,\hat{e}_2 - \sin\varphi\,\omega_{12}(\hat{T})\,\hat{e}_1$$

$$= (\omega_{12} + d\varphi)(\hat{T})\;J(\hat{T})$$

$$= \underbrace{\kappa_g}_{\text{geodesic curvature}} J(\hat{T})$$

$$\Rightarrow \boxed{\kappa_g = \underbrace{\frac{d\varphi}{ds}}_{\substack{\text{rotation}\\ \text{w.r.t.}\\ \text{frame}}} + \underbrace{\omega_{12}(\hat{T})}_{\substack{\text{rotation}\\ \text{of}\\ \text{frame}}}}$$

$$\therefore\ \nabla_{\vec{v}}\vec{v} = \nabla_{\vec{v}}\,v\hat{T} = \vec{v}[v]\,\hat{T} + v\,\nabla_{\vec{v}}\hat{T}$$

$$= \frac{dv}{dt}\hat{T} + \kappa_g v^2 J(\hat{T})$$

$\therefore$ $\boxed{\text{geodesic} \Leftrightarrow \kappa_g = 0 \ \&\ \text{const speed}}$

(pregeodesic $\Leftrightarrow \kappa_g = 0$)

Comparison with 3d

$$\hat{e}_1,\ \hat{e}_2,\ \hat{e}_3 = \hat{e}_1 \times \hat{e}_2 = \hat{n}$$

$$\Rightarrow \bar{\nabla}_{\vec{v}}\, \hat{e}_1 = \omega_{12}(\vec{v})\, \hat{e}_2 + \omega_{13}(\vec{v})\, \hat{n}$$

but $\omega_{13}(\vec{v}) = \bar{\nabla}_{\vec{v}}\, \hat{e}_1 \cdot \hat{n} = -\hat{e}_1 \cdot \bar{\nabla}_{\vec{v}}\, \hat{n}$

$$= +\,\hat{e}_1 \cdot S(\vec{v})$$

$$\therefore \boxed{\underbrace{\bar{\nabla}_{\vec{v}}}_{3d} \vec{w} = \underbrace{\nabla_{\vec{v}}}_{2d} \vec{w} + \left(\vec{w} \cdot S(\vec{v})\right) \hat{n}}$$