Total Curvature

Def: The total curvature of M is

$$\boxed{K_{tot} = \int K \, dM}$$

(M must be oriented for dM to exist globally)

Ex: sphere

$$K = \frac{1}{r^2} \Rightarrow K_{tot} = K \cdot \text{area} = \frac{1}{r^2} \cdot 4\pi r^2 = \underline{4\pi}$$

pseudosphere

$$K = -\frac{1}{\rho^2} \Rightarrow K_{tot} = K \cdot \text{area} = -\frac{1}{\rho^2} \cdot 2\pi \rho^2 = \underline{-2\pi}$$

catenoid

$$K = -\frac{1}{c^2 \cosh^4 z/c}, \quad dM = c\cosh^2 \frac{z}{c}\, dz \wedge d\theta \qquad \frac{d \tanh u}{du} = \frac{\cosh^2 u - \sinh^2 u}{\cosh^2 u}$$

$$\Rightarrow \int_C K \, dM = -\int_0^{2\pi}\int_{-a}^{a} \frac{1}{c\cosh^2 z/c}\, dz\, d\theta = -2\pi \tanh z/c \Big|_{-a}^{a} \to \underline{-4\pi}$$

helicoid

$$K = -\frac{b^2}{b^2 + r^2}, \quad dM = \sqrt{b^2 + r^2}\, dr\, d\theta$$

$$\Rightarrow \int_H K \, dM = -\int_0^{2\pi}\int_0^{a} \frac{b^2}{\sqrt{b^2 + r^2}}\, dr\, d\theta \qquad r = b\sinh\alpha \Rightarrow dr = b\cosh\alpha\, d\alpha = b\sqrt{1+\sinh^2\alpha}\, d\alpha$$

for 1 turn

$$= -2\pi b^2 \int_0^a \frac{dr}{\sqrt{b^2 + r^2}} = -2\pi b^2 \sinh^{-1}(r/b)\Big|_0^a \to \underline{\infty}$$

torus

$$K = \frac{\cos\theta}{\rho(R + \rho\cos\theta)}, \quad dM = \rho(R + \rho\cos\theta)\, d\theta\, d\phi$$

$$\Rightarrow \int_T K \, dM = \int_0^{2\pi}\int_0^{2\pi} \cos\theta\, d\theta\, d\phi = \underline{0}$$