<u>Recall</u>

$\{\hat{e}_i\}$ principal $\Rightarrow$

$$\boxed{\begin{aligned}\hat{e}_1[k_2] &= (k_1 - k_2)\,\omega_{12}(\hat{e}_2)\\ \hat{e}_2[k_1] &= (k_1 - k_2)\,\omega_{12}(\hat{e}_1)\end{aligned}}$$

convention: $k_1 \geq k_2$

(I) $S' \equiv 0 \Rightarrow$ planar

<u>Pf</u>: $S' = 0 \Rightarrow \hat{n} = \text{const}$

$\forall p, q \in M,\ \exists \alpha:\ \alpha(0) = p$
$\alpha(1) = q$

Let $f(u) = (\alpha(u) - p)\cdot\hat{n}$

$\Rightarrow \frac{df}{du} = \frac{d\alpha}{du}\cdot\hat{n} = 0$

$\Rightarrow f = f(0) = 0$

$\therefore f(1) = (q - p)\cdot\hat{n} = 0$

(II) $k_1 = k_2 \Rightarrow K = \text{const} \geq 0$

<u>Pf</u>: $k_1 = k_2 = k \Rightarrow \hat{e}_1[k] = 0 = \hat{e}_2[k]$

$\Rightarrow dk(\hat{e}_1) = 0 = dk(\hat{e}_2)$

$\Rightarrow dk = 0$

$\Rightarrow dK = d(k^2) = 2k\,dk = 0$

& $K = k^2 \geq 0$

(III) $k_1 = k_2$, $K > 0 \Rightarrow$ spherical

Pf: pick $p \in M$ & let $\vec{c} = \vec{p} + \frac{1}{k_p}\hat{n}_p$

as before, $\alpha(0) = p$

$\alpha(1) = q$

now let $\gamma = \alpha + \frac{1}{k}\hat{n}$

(II) $\Rightarrow K = \text{const} \Rightarrow k = \text{const}$

$$\Rightarrow \frac{d\gamma}{du} = \frac{d\alpha}{du} + \frac{1}{k}\frac{d\hat{n}}{du}$$

$$= \frac{d\alpha}{du} - \frac{1}{k}S\left(\frac{d\alpha}{du}\right)$$

$$= \frac{d\alpha}{du} - \frac{1}{k}k\frac{d\alpha}{du} = 0$$

$$\Rightarrow c = \gamma(0) = \gamma(1) = \vec{q} + \frac{1}{k}\hat{n}_q$$

$$\Rightarrow |\vec{q} - \vec{c}| = |1/k| = \text{const}$$

(IV) compact $\Rightarrow K > 0$ somewhere

Pf: $f = |\vec{p}|^2 \Rightarrow \exists\, p$ where $f = f_{max}$

Idea: M must be tangent to sphere of radius $|\vec{p}|$ at p

Let α be unit speed, $\alpha(0) = p$, $\hat{u} = \frac{d\alpha}{du}\Big|_p$

$\therefore f \circ \alpha = \vec{\alpha} \cdot \vec{\alpha}$

$$\& \quad \frac{d(\vec{\alpha}\cdot\vec{\alpha})}{du}\Big|_p = 0, \quad \frac{d^2(\vec{\alpha}\cdot\vec{\alpha})}{du^2}\Big|_p \leq 0$$

$$\therefore \quad \vec{\alpha} \cdot \hat{u} = 0, \text{ i.e. } \vec{p} \cdot \hat{u} = 0$$

$$\Longrightarrow \vec{p} \perp M \qquad \vec{p} = r\hat{n}$$

$$\therefore \quad \hat{u} \cdot \hat{u} + \vec{\alpha} \cdot \frac{d\hat{u}}{du}\Big|_p \leq 0$$

$$\Longrightarrow 1 + \underbrace{\vec{p} \cdot \frac{d\hat{u}}{du}\Big|_p}_{r \cdot \text{ normal curvature!}} \leq 0$$

$$\Rightarrow k(\hat{u}) \leq -\frac{1}{r}$$

$$\Rightarrow k_1 k_2 \geq \frac{1}{r^2} > 0$$

(V) $k_1|_p$ = local max, $k_2|_p$ = local min, $k_1 > k_2$

$\Rightarrow K \le 0$

Pf: local extrema $\Rightarrow \hat{e}_1[k_2] = 0 = \hat{e}_2[k_1]$ at p

$\hat{e}_1[\hat{e}_1[k_2]] \geq 0, \quad \hat{e}_2[\hat{e}_2[k_1]] \le 0$

$\Rightarrow \omega_{12}(\hat{e}_1) = 0 = \omega_{12}(\hat{e}_2) = 0$ at p

§6.2:2 $\Rightarrow K = \hat{e}_2[\omega_{12}(\hat{e}_1)] - \hat{e}_1[\omega_{12}(\hat{e}_2)]$

but $\hat{e}_1[k_2] = (k_1 - k_2)\omega_{12}(\hat{e}_2)$

$\Rightarrow \hat{e}_1[\hat{e}_1[k_2]] = 0 + (k_1 - k_2)\,\hat{e}_1[\omega_{12}(\hat{e}_2)] \geq 0$

& $\hat{e}_2[k_1] = (k_1 - k_2)\omega_{12}(\hat{e}_1)$

$\Rightarrow \hat{e}_2[\hat{e}_2[k_1]] = 0 + (k_1 - k_2)\,\hat{e}_2[\omega_{12}(\hat{e}_1)] \le 0$

$\Rightarrow K \le 0$ at p

(VI) compact, K = const $\Rightarrow$ sphere

Pf: compact $\Rightarrow \exists$ pt with k_1 = max

K = const $\Rightarrow k_2$ = min there

$k_1 > k_2 \Rightarrow K \le 0$ $\boxed{*}$

$\Rightarrow k_1 \equiv k_2 \Rightarrow$ sphere