# Special Curves

$\vec{\alpha}$ is <u>principal</u> $\Leftrightarrow \frac{d\vec{\alpha}}{du} \parallel \frac{d\hat{n}}{du}$

$\vec{\alpha}$ is <u>asymptotic</u> $\Leftrightarrow \frac{d^2\vec{\alpha}}{du^2} \perp \hat{n}$

$\vec{\alpha}$ is a <u>geodesic</u> $\Leftrightarrow \frac{d^2\vec{\alpha}}{du^2} \parallel \hat{n}$

<u>Recall</u>: $\hat{n} \cdot \frac{d\vec{\alpha}}{du} = 0 \Rightarrow \frac{d\hat{n}}{du} \cdot \frac{d\vec{\alpha}}{du} = -\hat{n} \cdot \frac{d^2\vec{\alpha}}{du^2}$

- $\vec{\alpha} = \underset{\hat{p}}{\text{plane}} \cap \underset{\hat{n}}{M}$ at constant angle $(\hat{n}\cdot\hat{p} = \text{const})$

$\Rightarrow \frac{d\vec{\alpha}}{du} \perp \hat{n}, \hat{p}$

$$\Downarrow$$

$$\frac{d\hat{n}}{du} \cdot \hat{p} = -\hat{n} \cdot \frac{d\hat{p}}{du} = 0$$

$$\Downarrow$$

$$\frac{d\hat{n}}{du} \perp \hat{n}, \hat{p}$$

$\Rightarrow \frac{d\vec{\alpha}}{du} \parallel \frac{d\hat{n}}{du}$

(or $\hat{n} = \pm\hat{p}$)

$\Rightarrow$ <u>principal</u>

<u>Ex</u>: latitude & longitude ("parallels" & "meridians") on <u>any</u> surface of revolution

- $\vec{\alpha} = \text{plane} \cap M$ at $\perp \Rightarrow \frac{d\vec{\alpha}}{ds} \perp \hat{p}$ as above

& $\vec{\alpha}$ unit speed (constant speed enough) $\quad \hat{\tau}$

$\Rightarrow \{\hat{\tau}, \hat{n}, \hat{p}\}$ is a frame

$\Rightarrow \frac{d\hat{\tau}}{ds} \cdot \hat{p} = -\hat{\tau} \cdot \frac{d\hat{p}}{du} = 0$

$\therefore \frac{d\hat{\tau}}{ds} \perp \hat{\tau}, \hat{p} \Rightarrow \frac{d^2\vec{\alpha}}{ds^2} = \frac{d\hat{\tau}}{ds} \parallel \hat{n}$

$\Rightarrow$ geodesic

<u>Ex</u>: all meridians (but not most parallels) on <u>any</u> surface of revolution

$\vec{\alpha}$ not constant speed $\Rightarrow$ <u>pregeodesic</u>

# Asymptotic Curves

- $k(\vec{v}) = k_1 \cos^2\theta + k_2 \sin^2\theta = 0$
  $\Rightarrow k_1 k_2 = 0 \Rightarrow$ M flat $\Rightarrow$ planar or $\exists!$ asymptotic direction (which is also principal!)
  OR
  $k_1 k_2 < 0 \Rightarrow \tan^2\theta = -\frac{k_1}{k_2} \Rightarrow \exists! 2$ asymptotic directions

- M minimal $\Rightarrow k_1 = -k_2$
  $\Rightarrow \tan^2\theta = 1$ (or planar)
  $\Rightarrow \theta = \pm\frac{\pi}{4} \Rightarrow \exists! 2$ ⊥ asymptotic directions