Revolution

Recall: surface of revolution

$$\vec{x} = r(u)\hat{r} + z(u)\hat{z}$$

$$\Rightarrow \boxed{\begin{aligned} \vec{e}_1 &= \frac{\partial \vec{x}}{\partial u} = \dot{r}\hat{r} + \dot{z}\hat{z} \\ \vec{e}_2 &= \frac{\partial \vec{x}}{\partial \theta} = r\hat{\theta} \end{aligned}}$$

where $\dot{r} = \frac{dr}{du}$, etc

since $\frac{d\hat{r}}{d\theta} = \hat{\theta}$

$$\Rightarrow \hat{n} = \frac{\vec{e}_1 \times \vec{e}_2}{|\vec{e}_1 \times \vec{e}_2|} = \frac{\dot{r}\hat{z} - \dot{z}\hat{r}}{\sqrt{\dot{r}^2 + \dot{z}^2}}$$

$$\Rightarrow \frac{\partial \hat{n}}{\partial u} = \frac{\ddot{r}\hat{z} - \ddot{z}\hat{r}}{\sqrt{\dot{r}^2 + \dot{z}^2}} - \frac{(\dot{r}\ddot{r} + \dot{z}\ddot{z})}{(\dot{r}^2 + \dot{z}^2)^{3/2}}(\dot{r}\hat{z} - \dot{z}\hat{r})$$

$$= \frac{(\dot{r}^2 + \dot{z}^2)(\ddot{r}\hat{z} - \ddot{z}\hat{r}) - (\dot{r}\ddot{r} + \dot{z}\ddot{z})(\dot{r}\hat{z} - \dot{z}\hat{r})}{(\dot{r}^2 + \dot{z}^2)^{3/2}}$$

$$= \frac{\hat{z}(\cancel{\dot{r}^2\ddot{r}} + \dot{z}^2\ddot{r} - \cancel{\dot{r}^2\ddot{r}} - \dot{r}\dot{z}\ddot{z}) + \hat{r}(-\dot{r}^2\ddot{z} - \cancel{\dot{z}^2\ddot{z}} + \dot{r}\dot{z}\ddot{r} + \cancel{\dot{z}\dot{z}\ddot{z}})}{(\dot{r}^2 + \dot{z}^2)^{3/2}}$$

$$= \frac{-(\dot{r}\hat{r} + \dot{z}\hat{z})(\dot{r}\ddot{z} - \dot{z}\ddot{r})}{(\dot{r}^2 + \dot{z}^2)^{3/2}} = \underbrace{\boxed{-\frac{\dot{r}\ddot{z} - \dot{z}\ddot{r}}{(\dot{r}^2 + \dot{z}^2)^{3/2}}}}_{\text{"}\,k_\mu}\vec{e}_1$$

$$\& \Rightarrow \frac{\partial \hat{n}}{\partial \theta} = -\frac{\dot{z}\hat{\theta}}{\sqrt{\dot{r}^2 + \dot{z}^2}} = \underbrace{\boxed{-\frac{1}{r}\frac{\dot{z}}{\sqrt{\dot{r}^2 + \dot{z}^2}}}}_{\text{"}\,k_\pi}\vec{e}_2$$

$$\Rightarrow \boxed{\begin{aligned} K &= \frac{\dot{z}}{r}\frac{\dot{r}\ddot{z} - \dot{z}\ddot{r}}{(\dot{r}^2 + \dot{z}^2)^2} \\ H &= -\frac{\dot{z}(\dot{r}^2 + \dot{z}^2) + r(\dot{r}\ddot{z} - \dot{z}\ddot{r})}{2r(\dot{r}^2 + \dot{z}^2)^{3/2}} \end{aligned}}$$

"canonical parametrization"

unit speed $\Rightarrow \dot{r}^2 + \dot{z}^2 = 1$

$$\Rightarrow \dot{r}\ddot{r} + \dot{z}\ddot{z} = 0$$

$$\Rightarrow \dot{r}\ddot{z} = -\frac{\dot{r}^2\ddot{r}}{\dot{z}}$$

$$\Rightarrow K = -\frac{(\dot{r}^2 + \dot{z}^2)\ddot{r}}{r(\dot{r}^2 + \dot{z}^2)^2} = \boxed{-\frac{\ddot{r}}{r}}$$