MTH 434 summary

curves $\vec{\alpha}(u)$

directional derivative: $\vec{v} = \frac{d\vec{\alpha}}{du} \Rightarrow \vec{v}[f] = \frac{df}{du}$

$$\nabla_{\vec{v}}\vec{w} = \frac{d\vec{w}}{du}$$

surfaces $\vec{x}(u,v)$

$$\vec{e}_1 = \frac{\partial \vec{x}}{\partial u}, \quad \vec{e}_2 = \frac{\partial \vec{x}}{\partial v}$$

$$\Rightarrow \vec{e}_1[f] = \frac{\partial f}{\partial u}$$

$$\nabla_{\vec{e}_1}\vec{w} = \frac{\partial \vec{w}}{\partial u}$$

orthogonal coords $\vec{e}_1 \perp \vec{e}_2$

normalize: $\hat{e}_i = \dfrac{\vec{e}_i}{|\vec{e}_i|}$

frame: $\{\hat{e}_1, \hat{e}_2, \hat{n} = \hat{e}_1 \times \hat{e}_2\}$

connection: $\nabla_{\vec{v}}\hat{e}_i = \omega_{ij}(\vec{v})\hat{e}_i$ $\quad \omega_{ji} = -\omega_{ij}$

shape operator: $S(\vec{v}) = -\nabla_{\vec{v}}\hat{n} = -\omega_{3i}(\vec{v})\hat{e}_i$ $\quad S^T = S$

$$\Rightarrow S = (\omega_{i3}(\hat{e}_j))$$

Structure equations

dual basis: $\sigma_i(\hat{e}_j) = \delta_{ij}$

1st:

$$d\sigma_1 = \omega_{12} \wedge \sigma_2$$
$$d\sigma_2 = \omega_{21} \wedge \sigma_1$$

$$\omega_{13} \wedge \sigma_1 + \omega_{23} \wedge \sigma_2 = 0$$

2nd:

$$d\omega_{12} = \omega_{13} \wedge \omega_{32} \quad \text{Gauss}$$
$$\left.\begin{aligned} d\omega_{13} &= \omega_{12} \wedge \omega_{23} \\ d\omega_{23} &= \omega_{21} \wedge \omega_{13} \end{aligned}\right\} \text{Codazzi}$$

Curvature

$$K = \det S \longleftrightarrow d\omega_{12} = -K\,\sigma_1 \wedge \sigma_2$$
$$2H = \operatorname{tr} S \longleftrightarrow \omega_{13} \wedge \sigma_2 + \sigma_1 \wedge \omega_{23} = 2H\,\sigma_1 \wedge \sigma_2$$