

§7.6:6

$$\begin{aligned} \textcircled{I} \quad \vec{v} = \dot{\vec{e}}_1 = h_1 \dot{\vec{e}}_1 &\Rightarrow J(\vec{v}) = h_1 J(\dot{\vec{e}}_1) = h_1 \dot{\vec{e}}_2 = \frac{h_1}{h_2} \vec{e}_2 \\ \Rightarrow \vec{a} \cdot J(\vec{v}) &= \frac{d\vec{e}_1}{du} \cdot \frac{h_1}{h_2} \vec{e}_2 = \frac{h_1}{h_2} \vec{x}_{uu} \cdot \vec{x}_v = -\frac{h_1}{h_2} \vec{x}_u \cdot \vec{x}_{uv} \\ &= -\frac{h_1}{h_2} \vec{x}_u \cdot \vec{x}_{uv} = -\frac{1}{2} \frac{h_1}{h_2} (\vec{x}_u \cdot \vec{x}_v)_v \\ &= -\frac{1}{2} \frac{h_1}{h_2} \frac{d(h_1^2)}{dv} = -\frac{1}{2} \sqrt{\frac{E}{G}} \frac{\partial E}{\partial v} \\ &= -\frac{h_1^2}{h_2} \frac{dh_1}{dv} \end{aligned}$$

$$\Rightarrow \frac{\vec{a} \cdot J(\vec{v})}{v^2} = -\frac{1}{h_2} \frac{dh_1}{dv} = -\frac{1}{2\sqrt{EG}} \frac{\partial E}{\partial v}$$

$$\begin{aligned} \textcircled{II} \quad \begin{matrix} \sigma_1 = h_1 du \\ \sigma_2 = h_2 dv \end{matrix} &\Rightarrow \begin{matrix} d\sigma_1 = dh_1 du = -\frac{\partial h_1}{\partial v} dh_2 dv = -\frac{1}{h_1 h_2} \frac{\partial h_1}{\partial v} \sigma_1 \sigma_2 \\ d\sigma_2 = dh_2 dv = \frac{\partial h_2}{\partial u} du dv = -\frac{1}{h_1 h_2} \frac{\partial h_2}{\partial u} \sigma_2 \sigma_1 \end{matrix} \\ \Rightarrow \omega_{12} &= \frac{1}{h_1 h_2} \left(\frac{\partial h_2}{\partial u} \sigma_2 - \frac{\partial h_1}{\partial v} \sigma_1 \right) \\ &= \frac{1}{h_1} \frac{\partial h_2}{\partial u} dv - \frac{1}{h_2} \frac{\partial h_1}{\partial v} du \\ \Rightarrow \omega_{12}(\vec{e}_1) &= -\frac{1}{h_2} \frac{\partial h_1}{\partial v} = -\frac{1}{2\sqrt{EG}} \frac{\partial E}{\partial v} \end{aligned}$$

$$\begin{aligned} \Rightarrow \vec{a} \cdot J(\vec{v}) &= \nabla_{\vec{e}_1} \vec{e}_1 \cdot \frac{h_1}{h_2} \vec{e}_2 \\ &= \nabla_{\vec{e}_1} h_1 \vec{e}_1 \cdot h_1 \vec{e}_2 \\ &= (h_1 \vec{e}_1 + h_1 \nabla_{\vec{e}_1} \vec{e}_1) \cdot h_1 \vec{e}_2 \\ &= h_1^2 \omega_{12}(\vec{e}_1) = v^2 \omega_{12}(\vec{e}_1) \end{aligned}$$

$$\Rightarrow \frac{\vec{a} \cdot J(\vec{v})}{v^2} = \omega_{12}(\vec{e}_1) = -\frac{1}{h_2} \frac{\partial h_1}{\partial v} = -\frac{1}{2\sqrt{EG}} \frac{\partial E}{\partial v}$$