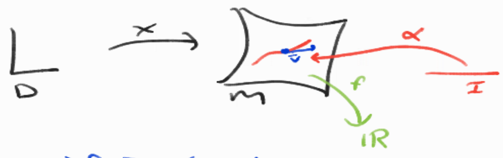
Geodesics

curve



$$\vec{v}[f] = (f \circ x)^{\cdot} = \text{"}\dot{f}\text{"}$$

Def: α geodesic $\Leftrightarrow \nabla_{\vec{v}} \vec{v} = 0$

Components

$$\vec{v} = v_i \vec{e}_i \qquad \leftarrow \text{frame}$$

$$\Rightarrow \nabla_{\vec{v}} \vec{v} = \nabla_{\vec{v}} v_i \vec{e}_i$$

$$\therefore \; 0 = \vec{v}[v_i] \vec{e}_i + v_i \omega_{ij}(\vec{v}) \vec{e}_j$$

$$\Rightarrow 0 = \vec{v}[v_j] + v_i \omega_{ij}(\vec{v})$$

$$\therefore \; \boxed{\begin{aligned} 0 &= \vec{v}[v_1] - \omega_{12}(\vec{v}) v_2 \\ 0 &= \vec{v}[v_2] + \omega_{12}(\vec{v}) v_1 \end{aligned}}$$

Orthogonal Coordinates

$$d\vec{x} = \frac{\partial \vec{x}}{\partial u}\,du + \frac{\partial \vec{x}}{\partial v}\,dv$$
$$= \vec{e}_1\,du + \vec{e}_2\,dv$$

$$\vec{e}_1 \perp \vec{e}_2 \Rightarrow d\vec{x}\cdot d\vec{x} = \vec{e}_1\cdot\vec{e}_1\,du^2 + \vec{e}_2\cdot\vec{e}_2\,dv^2$$
$$\Rightarrow \sigma_1 = |\vec{e}_1|\,du = \sqrt{E}\,du = h_1\,du$$
$$\sigma_2 = |\vec{e}_2|\,dv = \sqrt{G}\,dv = h_2\,dv$$
$$\Rightarrow ds^2 = h_1^2\,du^2 + h_2^2\,dv^2$$

$$\therefore d\sigma_1 = \frac{\partial h_1}{\partial v}\,dv\wedge du = \omega_{12}\wedge h_2\,dv$$
$$d\sigma_2 = \frac{\partial h_2}{\partial u}\,du\wedge dv = -\omega_{12}\wedge h_1\,du$$

$$\Rightarrow \boxed{\omega_{12} = -\frac{1}{h_2}\frac{\partial h_1}{\partial v}\,du + \frac{1}{h_1}\frac{\partial h_2}{\partial u}\,dv}$$
$$= \frac{1}{h_1 h_2}\left(-\frac{\partial h_1}{\partial v}\sigma_1 + \frac{\partial h_2}{\partial u}\sigma_2\right)$$

$$\vec{v} = \dot{\alpha}_1\vec{e}_1 + \dot{\alpha}_2\vec{e}_2$$ ← coord basis
$$= h_1\dot{\alpha}_1\hat{e}_1 + h_2\dot{\alpha}_2\hat{e}_2$$ ← frame
$$\Rightarrow \omega_{12}(\vec{v}) = -\frac{\dot{\alpha}_1}{h_2}\frac{\partial h_1}{\partial v} + \frac{\dot{\alpha}_2}{h_1}\frac{\partial h_2}{\partial u}$$

$$\therefore v_1 = h_1\dot{\alpha}_1,\quad v_2 = h_2\dot{\alpha}_2$$
$$\Rightarrow \vec{v}[v_1] = (h_1\dot{\alpha}_1)^{\cdot} = h_1\ddot{\alpha}_1 + \frac{\partial h_1}{\partial u}\dot{\alpha}_1^2 + \frac{\partial h_1}{\partial v}\dot{\alpha}_1\dot{\alpha}_2$$
$$\Rightarrow 0 = \vec{v}[v_1] - \omega_{12}(\vec{v})\,v_2$$
$$= (h_1\dot{\alpha}_1)^{\cdot} - \omega_{12}(\vec{v})(h_2\dot{\alpha}_2)$$
$$= (h_1\dot{\alpha}_1)^{\cdot} + \dot{\alpha}_1\dot{\alpha}_2\frac{\partial h_1}{\partial v} - \frac{h_2}{h_1}\dot{\alpha}_2^2\frac{\partial h_2}{\partial u}$$

$$\Rightarrow \boxed{\ddot{\alpha}_1 + \frac{1}{h_1}\frac{\partial h_1}{\partial u}\dot{\alpha}_1^2 + \frac{2}{h_1}\frac{\partial h_1}{\partial v}\dot{\alpha}_1\dot{\alpha}_2 - \frac{h_2}{h_1^2}\frac{\partial h_2}{\partial u}\dot{\alpha}_2^2 = 0}$$

similarly,
$$\vec{v}[v_2] = (h_2\dot{\alpha}_2)^{\cdot}$$
$$\Rightarrow 0 = (h_2\dot{\alpha}_2)^{\cdot} + \omega_{12}(\vec{v})(h_1\dot{\alpha}_1)$$
$$= (h_2\dot{\alpha}_2)^{\cdot} - \frac{h_1}{h_2}\dot{\alpha}_1^2\frac{\partial h_1}{\partial v} + \dot{\alpha}_1\dot{\alpha}_2\frac{\partial h_2}{\partial u}$$

$$\Rightarrow \boxed{\ddot{\alpha}_2 + \frac{1}{h_2}\frac{\partial h_2}{\partial v}\dot{\alpha}_2^2 + \frac{2}{h_2}\frac{\partial h_2}{\partial u}\dot{\alpha}_1\dot{\alpha}_2 - \frac{h_1}{h_2^2}\frac{\partial h_1}{\partial v}\dot{\alpha}_1^2 = 0}$$

coupled 2nd order ODEs

$\therefore$ through every point in every direction $\exists!$ geodesic