Adapted Frame Fields

An *adapted frame* on M is:

$\hat{e}_1 \perp \hat{e}_2 \in TM$

$\hat{e}_3 = \hat{e}_1 \times \hat{e}_2 =: \hat{n}$

$$\Rightarrow \quad S = \begin{pmatrix} \omega_{13}(\hat{e}_1) & \omega_{23}(\hat{e}_1) \\ \omega_{13}(\hat{e}_2) & \omega_{23}(\hat{e}_2) \end{pmatrix} \text{ in this basis}$$

Dual basis *in* $\mathbb{R}^3$:

$$\sigma_i(\hat{e}_j) = \delta_{ij}$$

$\Rightarrow \sigma_3(\vec{v}) = 0 \Rightarrow \sigma_3 = 0$ "on" M! (pullback of $\sigma_3 = 0$)

$\therefore$ on M:

$\{\hat{e}_1, \hat{e}_2\}$ frame

$\{\sigma_1, \sigma_2\}$ dual basis

$\omega_{12}, \omega_{13}, \omega_{23}$ connection (with σ_3 set to 0)

Structure equations

In $\mathbb{R}^3$:

$$d\sigma_i = \omega_{ij} \wedge \sigma_j$$

$$d\omega_{ij} = \omega_{ik} \wedge \omega_{kj}$$

$\therefore$ on M:

$$d\sigma_1 = \omega_{12} \wedge \sigma_2 + 0$$

$$d\sigma_2 = -\omega_{12} \wedge \sigma_1 + 0$$

$$0 = d\sigma_3 = \omega_{31} \wedge \sigma_1 + \omega_{32} \wedge \sigma_2 \quad \text{Symmetry of } S!$$

$$d\omega_{12} = \omega_{13} \wedge \omega_{32} = -\omega_{13} \wedge \omega_{23} \quad \text{Gauss}$$

$$\left.\begin{aligned} d\omega_{13} &= \omega_{12} \wedge \omega_{23} \\ d\omega_{23} &= \omega_{21} \wedge \omega_{13} = -\omega_{12} \wedge \omega_{13} \end{aligned}\right\} \text{Codazzi}$$

Curvature

$$\omega_{13} \wedge \omega_{23} = K\, \sigma_1 \wedge \sigma_2$$

$$\omega_{13} \wedge \sigma_2 + \sigma_1 \wedge \omega_{23} = 2H\, \sigma_1 \wedge \sigma_2$$

$$\boxed{d\omega_{12} = -K\, \sigma_1 \wedge \sigma_2}$$

Theorema Egregium (Gauss)

Principal Frame Fields

A <u>principal frame</u> field is an adapted frame field such that $\hat{e}_1, \hat{e}_2$ are principal

Example: surface of revolution

non-umbilic $\Rightarrow$ ① $\exists$ principal frame in nbhd
② determined up to sign

adapted $\Rightarrow S = \begin{pmatrix} \omega_{13}(\hat{e}_1) & \omega_{23}(\hat{e}_1) \\ \omega_{13}(\hat{e}_2) & \omega_{23}(\hat{e}_2) \end{pmatrix}$

principal $\Rightarrow \omega_{13}(\hat{e}_1) = k_1, \ \omega_{23}(\hat{e}_2) = k_2$
$\omega_{13}(\hat{e}_2) = 0, \ \omega_{23}(\hat{e}_1) = 0$

$$\Rightarrow \boxed{\begin{array}{l} \omega_{13} = k_1 \sigma_1 \\ \omega_{23} = k_2 \sigma_2 \end{array}}$$

$$\Rightarrow d\omega_{13} = \omega_{12} \wedge \omega_{23} = \omega_{12} \wedge k_2 \sigma_2$$

$$d\omega_{13} = dk_1 \wedge \sigma_1 + k_1 d\sigma_1 = dk_1 \wedge \sigma_1 + k_1 \omega_{12} \wedge \sigma_2$$

$$\Rightarrow dk_1 \wedge \sigma_1 = (k_2 - k_1)\, \omega_{12} \wedge \sigma_2$$

Recall: $\alpha \wedge \beta (\hat{e}_1, \hat{e}_2) = \begin{vmatrix} \alpha(\hat{e}_1) & \alpha(\hat{e}_2) \\ \beta(\hat{e}_1) & \beta(\hat{e}_2) \end{vmatrix}$

$$\Rightarrow \boxed{\hat{e}_2[k_1] = dk_1(\hat{e}_2) = (k_1 - k_2)\, \omega_{12}(\hat{e}_1)}$$

Similarly,

$$\boxed{\hat{e}_1[k_2] = dk_2(\hat{e}_1) = (k_1 - k_2)\, \omega_{12}(\hat{e}_2)}$$

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① $df = \frac{\partial f}{\partial u_i} du_i$

② $df(\hat{e}_i) = \hat{e}_i[f]$

$\Rightarrow df = \hat{e}_i[f]\, \sigma_i$