Metric Compatibility

$$\nabla_{\vec{v}} \vec{p} = \vec{v}[p_i]\,\hat{e}_i + p_i\,\omega_{ij}(\vec{v})\,\hat{e}_j$$

$$\Rightarrow \nabla_{\vec{v}} \vec{p} \cdot \vec{q} = \vec{v}[p_i]\,q_i + \cancel{p_i q_j \omega_{ij}}(\vec{v})$$

$$\vec{p} \cdot \nabla_{\vec{v}} \vec{q} = p_i\,\vec{v}[q_i] + \cancel{q_i p_j \omega_{ij}}(\vec{v})$$

$$\boxed{\langle \nabla_{\vec{v}} \vec{p}, \vec{q} \rangle + \langle \vec{p}, \nabla_{\vec{v}} \vec{q} \rangle = \vec{v}[\langle \vec{p}, \vec{q} \rangle]} + 0$$

$$\Rightarrow \vec{v}[\underbrace{|\vec{p}|^2}_{\langle \vec{p}, \vec{p} \rangle}] = 2\,\nabla_{\vec{v}} \vec{p} \cdot \vec{p}$$

Velocity

In $\mathbb{R}^3$ $\quad \alpha \longmapsto \vec{\alpha} \longrightarrow \vec{v} = \frac{d\vec{\alpha}}{du}$

doesn't exist!

$\therefore$ curve $\quad \alpha : I \to M \xrightarrow{f} \mathbb{R}$

velocity $\quad \vec{v}[f] = \frac{d(f \circ \alpha)}{du}$

$\therefore \vec{v} = $ "$\frac{d}{du}$"

Book : $\vec{v} = \alpha'$

$\therefore$ write $\vec{v} = \frac{d\vec{\alpha}}{du}$

Parallel Vectors

Def: $\vec{w}$ is parallel along α $\iff \nabla_{\vec{v}} \vec{w} = 0$, where $\vec{v} = \frac{d\vec{\alpha}}{du}$

" $\vec{w}$ $\parallel$ $\iff \vec{w}' = 0$ "

$\vec{w} \parallel \Rightarrow \vec{v}[|\vec{w}|^2] = \vec{v}[\langle \vec{w}, \vec{w} \rangle] = 2 \cancel{\nabla_{\vec{v}}} \vec{w} \cdot \vec{w} = 0$

$\Rightarrow |\vec{w}| = \text{const}$

$\Rightarrow \vec{w} = c \, (\hat{e}_1 \cos\varphi + \hat{e}_2 \sin\varphi)$

$\Rightarrow \nabla_{\vec{v}} (\hat{e}_1 \cos\varphi + \hat{e}_2 \sin\varphi) = 0$

$\Rightarrow \omega_{12}(\vec{v})(\hat{e}_2 \cos\varphi - \hat{e}_1 \sin\varphi)$
$\quad + (-\hat{e}_1 \sin\varphi + \hat{e}_2 \cos\varphi) \vec{v}[\varphi] = 0$

$\Rightarrow \vec{v}[\varphi] + \omega_{12}(\vec{v}) = 0$

$\Rightarrow (d\varphi + \omega_{12})(\vec{v}) = 0$

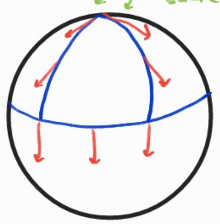
"$d\varphi = -\omega_{12}$ along curve"

$\therefore$ holonomy angle "$\Delta\varphi$"

$$\boxed{\psi = -\oint \omega_{12}}$$

Holonomy

- Parallel transport a vector around a triangle on $\mathbb{S}^2$



Ex: $\mathbb{S}^2$, $ds^2 = r^2(d\theta^2 + \sin^2\theta \, d\varphi^2)$

$$\sigma_1 = r\,d\theta, \quad \sigma_2 = r\sin\theta\, d\phi$$

$$\Rightarrow \omega_{12} = \cos\theta\, d\phi$$

$$\Rightarrow \Psi = -\int_0^{2\pi} \cos\theta\, d\phi$$

$$= -2\pi\cos\theta$$

Foucault pendulum!
(clockwise in North
counterclockwise in South)