Connection

$\{\hat{f}_1, \hat{f}_2\}$ a frame

$$\Rightarrow \quad \hat{f}_1 = \hat{e}_1 \cos\alpha + \hat{e}_2 \sin\alpha$$
$$\hat{f}_2 = -\hat{e}_1 \sin\alpha + \hat{e}_2 \cos\alpha$$

$$\Rightarrow \quad \tau_1 = \sigma_1 \cos\alpha + \sigma_2 \sin\alpha$$
$$\tau_2 = -\sigma_1 \sin\alpha + \sigma_2 \cos\alpha$$

$$\Rightarrow \quad d\tau_1 = d\sigma_1 \cos\alpha + d\sigma_2 \sin\alpha + d\alpha \wedge (-\sigma_1 \sin\alpha + \sigma_2 \cos\alpha)$$
$$= \omega_{12} \wedge \sigma_2 \cos\alpha - \omega_{12} \wedge \sigma_1 \sin\alpha + d\alpha \wedge \tau_2$$
$$= (\omega_{12} + d\alpha) \wedge \tau_2$$

$$d\tau_2 = -d\sigma_1 \sin\alpha + d\sigma_2 \cos\alpha + d\alpha \wedge (-\sigma_1 \cos\alpha - \sigma_2 \sin\alpha)$$
$$= -\omega_{12} \wedge \sigma_2 \sin\alpha - \omega_{12} \wedge \sigma_1 \cos\alpha - d\alpha \wedge \tau_1$$
$$= -(\omega_{12} + d\alpha) \wedge \tau_1$$

$$\therefore \boxed{\omega_{12} \longmapsto \omega_{12} + d\alpha}$$

Covariant Derivative

Define $\boxed{\nabla_{\vec{v}} \vec{w} = \vec{v}[w_i]\,\hat{e}_i + w_i\, \omega_{ij}(\vec{v})\,\hat{e}_j}$

Check well-defined?

$$\hat{f}_1 = \hat{e}_1 \cos\alpha + \hat{e}_2 \sin\alpha$$
$$\hat{f}_2 = -\hat{e}_1 \sin\alpha + \hat{e}_2 \cos\alpha$$

$$\begin{aligned}
\nabla_{\vec{v}} \hat{f}_1 &= \nabla_{\vec{v}} \left(\hat{e}_1 \cos\alpha + \hat{e}_2 \sin\alpha\right) \\
&= \cos\alpha \nabla_{\vec{v}} \hat{e}_1 + \sin\alpha \nabla_{\vec{v}} \hat{e}_2 + \left(-\hat{e}_1 \sin\alpha + \hat{e}_2 \cos\alpha\right) \vec{v}[\alpha] \\
&= \omega_{12}(\vec{v}) \left(\cos\alpha\, \hat{e}_2 - \sin\alpha\, \hat{e}_1\right) + \left(-e_1 \sin\alpha + \hat{e}_2 \cos\alpha\right) d\alpha(\vec{v}) \\
&= \left(\omega_{12}(\vec{v}) + d\alpha(\vec{v})\right) \hat{f}_2 = \bar{\omega}_{12}(\vec{v})\, \hat{f}_2
\end{aligned}$$