

Clairaut Parametrization

$$ds^2 = h_1^2 du^2 + h_2^2 dv^2$$

$h_1 = h_1(u), h_2 = h_2(v)$

$$\omega_{12} = -\frac{1}{h_2} \frac{\partial h_1}{\partial v} du + \frac{1}{h_1} \frac{\partial h_2}{\partial u} dv$$
$$= \frac{1}{h_1 h_2} \left(-\frac{\partial h_1}{\partial v} \sigma_1 + \frac{\partial h_2}{\partial u} \sigma_2 \right)$$

$$\ddot{u} + \frac{1}{h_1} \frac{\partial h_1}{\partial u} \dot{u}^2 + \frac{2}{h_1} \frac{\partial h_1}{\partial v} \dot{u} \dot{v} - \frac{h_2}{h_1^2} \frac{\partial h_2}{\partial u} \dot{v}^2 = 0$$
$$\ddot{v} + \frac{1}{h_2} \frac{\partial h_2}{\partial v} \dot{v}^2 + \frac{2}{h_2} \frac{\partial h_2}{\partial u} \dot{u} \dot{v} - \frac{h_1}{h_2^2} \frac{\partial h_1}{\partial v} \dot{u}^2 = 0$$

$$\omega_{12} = \frac{1}{h_1} \frac{\partial h_2}{\partial u} dv = \frac{1}{h_1 h_2} \frac{\partial h_2}{\partial u} \sigma_2$$

$$\ddot{u} + \frac{1}{h_1} \frac{\partial h_1}{\partial u} \dot{u}^2 - \frac{h_2}{h_1^2} \frac{\partial h_2}{\partial u} \dot{v}^2 = 0$$
$$\ddot{v} + \frac{2}{h_2} \frac{\partial h_2}{\partial u} \dot{u} \dot{v} = 0$$

Ex: meridians

$v = \text{const}; h_1 \dot{u} = 1$ ← unit speed

$$\Rightarrow \frac{\partial h_1}{\partial u} \dot{u}^2 + h_1 \ddot{u} = 0 \checkmark$$

∴ geodesic

Examples

$$\begin{aligned} u &= \alpha_1(t) \\ v &= \alpha_2(t) \end{aligned} \quad \mapsto$$

$$\begin{aligned} \ddot{u} + \frac{1}{h_1} \frac{\partial h_1}{\partial u} \dot{u}^2 + \frac{2}{h_1} \frac{\partial h_1}{\partial v} \dot{u} \dot{v} - \frac{h_2}{h_1^2} \frac{\partial h_2}{\partial u} \dot{v}^2 &= 0 \\ \ddot{v} + \frac{1}{h_2} \frac{\partial h_2}{\partial v} \dot{v}^2 + \frac{2}{h_2} \frac{\partial h_2}{\partial u} \dot{u} \dot{v} - \frac{h_1}{h_2^2} \frac{\partial h_1}{\partial v} \dot{u}^2 &= 0 \end{aligned}$$

① rectangular coords

$$\begin{aligned} h_1 &= 1 \\ h_2 &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \ddot{x} &= 0 \\ \ddot{y} &= 0 \end{aligned}$$

② polar coords

$$\begin{aligned} h_1 &= 1 \\ h_2 &= r \end{aligned} \quad \Rightarrow \quad \begin{aligned} \ddot{r} - r \dot{\theta}^2 &= 0 \\ \ddot{\theta} + \frac{2}{r} \dot{r} \dot{\theta} &= 0 \end{aligned}$$

③ sphere

$$\begin{aligned} h_1 &= r \\ h_2 &= r \sin \theta \end{aligned} \quad \Rightarrow \quad \begin{aligned} \ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 &= 0 \\ \ddot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} &= 0 \end{aligned}$$