Clairaut Parametrization

$$ds^2 = h_1^2\,du^2 + h_2^2\,dv^2$$

$$h_1 = h_1(u),\quad h_2 = h_2(v)$$

$$\omega_{12} = -\frac{1}{h_2}\frac{\partial h_1}{\partial v}\,du + \frac{1}{h_1}\frac{\partial h_2}{\partial u}\,dv$$

$$= \frac{1}{h_1 h_2}\left(-\frac{\partial h_1}{\partial v}\sigma_1 + \frac{\partial h_2}{\partial u}\sigma_2\right)$$

$$\ddot{u} + \frac{1}{h_1}\frac{\partial h_1}{\partial u}\dot{u}^2 + \frac{2}{h_1}\frac{\partial h_1}{\partial v}\dot{u}\dot{v} - \frac{h_2}{h_1^2}\frac{\partial h_2}{\partial u}\dot{v}^2 = 0$$

$$\ddot{v} + \frac{1}{h_2}\frac{\partial h_2}{\partial v}\dot{v}^2 + \frac{2}{h_2}\frac{\partial h_2}{\partial u}\dot{u}\dot{v} - \frac{h_1}{h_2}\frac{\partial h_1}{\partial v}\dot{u}^2 = 0$$

$$\omega_{12} = \frac{1}{h_1}\frac{\partial h_2}{\partial u}\,dv = \frac{1}{h_1 h_2}\frac{\partial h_2}{\partial u}\sigma_2$$

$$\ddot{u} + \frac{1}{h_1}\frac{\partial h_1}{\partial u}\dot{u}^2 - \frac{h_2}{h_1^2}\frac{\partial h_2}{\partial u}\dot{v}^2 = 0$$

$$\ddot{v} + \frac{2}{h_2}\frac{\partial h_2}{\partial u}\dot{u}\dot{v} = 0$$

Ex: meridians

$v = \text{const}$; $h_1\dot{u} = 1$ ← unit speed

$$\Rightarrow \frac{\partial h_1}{\partial u}\dot{u}^2 + h_1\ddot{u} = 0 \;\checkmark$$

$\therefore$ geodesic

Examples

$u = \alpha_1(t)$
$v = \alpha_2(t)$ $\longmapsto$

$$\ddot{u} + \frac{1}{h_1}\frac{\partial h_1}{\partial u}\dot{u}^2 + \frac{2}{h_1}\frac{\partial h_1}{\partial v}\dot{u}\dot{v} - \frac{h_2}{h_1^2}\frac{\partial h_2}{\partial u}\dot{v}^2 = 0$$
$$\ddot{v} + \frac{1}{h_2}\frac{\partial h_2}{\partial v}\dot{v}^2 + \frac{2}{h_2}\frac{\partial h_2}{\partial u}\dot{u}\dot{v} - \frac{h_1}{h_2^2}\frac{\partial h_1}{\partial v}\dot{u}^2 = 0$$

① rectangular coords

$h_1 = 1$
$h_2 = 1$ $\Rightarrow$ $\ddot{x} = 0$, $\ddot{y} = 0$

② polar coords

$h_1 = 1$
$h_2 = r$ $\Rightarrow$ $\ddot{r} - r\dot{\theta}^2 = 0$, $\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} = 0$

③ sphere

$h_1 = r$
$h_2 = r\sin\theta$ $\Rightarrow$ $\ddot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0$, $\ddot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi} = 0$