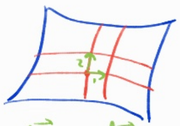
Area

Idea



$$d\vec{A} = d\vec{x}_1 \times d\vec{x}_2 = \vec{e}_1 \times \vec{e}_2 \, du\, dv$$

Recall
$$|\vec{e}_1 \times \vec{e}_2|^2 = |\vec{e}_1|^2 |\vec{e}_2|^2 \sin^2\theta$$
$$= |\vec{e}_1|^2 |\vec{e}_2|^2 (1-\cos^2\theta)$$
$$= (\vec{e}_1\cdot\vec{e}_1)(\vec{e}_2\cdot\vec{e}_2) - (\vec{e}_1\cdot\vec{e}_2)^2$$
$$= EG - F^2 = \det g$$

Def: area form is a 2-form "dM" such that
$$dM(\vec{e}_1, \vec{e}_2) = \pm\sqrt{\det g}$$

Equivalently, $dM(\hat{e}_1, \hat{e}_2) = \pm 1$ for any frame
$$\Rightarrow dM = \pm\, \sigma_1 \wedge \sigma_2$$
locally: 2 choices
globally: need M orientable

Equivalently, $dM(\vec{u}, \vec{v}) = \pm\sqrt{(\vec{u}\cdot\vec{u})(\vec{v}\cdot\vec{v}) - (\vec{u}\cdot\vec{v})^2}$
$$= \pm|\vec{u}\times\vec{v}| = \pm\, \hat{n}\cdot\vec{u}\times\vec{v}$$

Ex: $\mathbb{R}^2 \to S^2$
$$(\theta, \phi) \mapsto r\sin\theta\cos\phi\,\hat{x} + r\sin\theta\sin\phi\,\hat{y} + r\cos\theta\,\hat{z}$$
$$\Rightarrow \vec{e}_1 = r\cos\theta\cos\phi\,\hat{x} + r\cos\theta\sin\phi\,\hat{y} - r\sin\theta\,\hat{z}$$
$$\vec{e}_2 = -r\sin\theta\sin\phi\,\hat{x} + r\sin\theta\cos\phi\,\hat{y}$$

dual basis is $\{d\theta, d\phi\}$

but area form dS must satisfy
$$dS(\vec{e}_1, \vec{e}_2) = \sqrt{|\vec{e}_i\cdot\vec{e}_j|} = r^2\sin\theta$$
$$\Leftrightarrow \underline{dS = r^2 \sin\theta\, d\theta \wedge d\phi}$$

OR:
$$d\vec{x}_1 = r\, d\theta\, \hat{\theta}$$
$$d\vec{x}_2 = r\sin\theta\, d\phi\, \hat{\phi}$$
$$\Rightarrow d\vec{A} = d\vec{x}_1 \times d\vec{x}_2 = r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$$

$$\Rightarrow A = \int_{S^2} dS = \int_0^{2\pi}\int_0^{\pi} r^2\sin\theta\, d\theta\, d\phi = 4\pi r^2$$

Pseudosphere

$$\Rightarrow d\vec{x} = \left[\left(-e^{-u}\hat{r} + \sqrt{1-e^{-2u}}\,\hat{z}\right)du + e^{-u}\hat{\theta}\,d\theta\right]\rho$$

$$\Rightarrow ds^2 = d\vec{x}\cdot d\vec{x} = \left(\underline{du^2 + e^{-2u}d\theta^2}\right)\rho^2$$

$$\& \Rightarrow d\vec{y} = \left[\left(\cos\alpha\,\hat{r} - \frac{\cos^2\alpha}{\sin\alpha}\hat{z}\right)d\alpha + \sin\alpha\,\hat{\theta}\,d\theta\right]\rho$$

$$\Rightarrow ds^2 = d\vec{y}\cdot d\vec{y} = \left[\left(\cos^2\alpha + \frac{\cos^4\alpha}{\sin^2\alpha}\right)d\alpha^2 + \sin^2\alpha\,d\theta^2\right]\rho^2$$

$$= \left(\underline{\cot^2\alpha\,d\alpha^2 + \sin^2\alpha\,d\theta^2}\right)\rho^2$$

$$\boxed{\begin{array}{l} \therefore\ \sigma_1 = \rho\,du,\ \sigma_2 = \rho e^{-u}d\theta \\ \therefore\ \sigma_1 = \rho\cot\alpha\,d\alpha,\ \sigma_2 = \rho\sin\alpha\,d\theta \end{array}}$$

$$\Rightarrow dM = \sigma_1\wedge\sigma_2 = \rho^2 e^{-u}du\wedge d\theta$$

$$= -\rho^2\cos\alpha\,d\alpha\wedge d\theta$$

$$\Rightarrow A = \int_M dM = \int_0^{2\pi}\int_0^{\infty}\rho^2 e^{-u}du\,d\theta = -2\pi\rho^2 e^{-u}\Big|_0^{\infty} = 2\pi\rho^2$$

$$= -\int_0^{2\pi}\int_0^{\pi/2}\rho^2\sin\alpha\,d\alpha\,d\theta = +2\pi\rho^2\cos\alpha\Big|_0^{\pi/2} = -2\pi\rho^2$$

Catenoid

$$r = c\cosh z/c \Rightarrow dr = \sinh z/c\,dz$$

$$\Rightarrow ds^2 = dr^2 + r^2d\theta^2 + dz^2$$

$$= (1+\sinh^2 z/c)\,dz^2 + c^2\cosh^2 z/c\,d\theta^2$$

$$= \cosh^2 z/c\,(dz^2 + c^2d\theta^2)$$

$$\Rightarrow dM = \cosh z/c\,dz\wedge c\cosh z/c\,d\theta$$

$$= c\cosh^2 z/c\,dz\wedge d\theta$$

$$\cosh 2z = \cosh^2 z + \sinh^2 z = 1 + 2\sinh^2 z = 2\cosh^2 z - 1$$

$$\Rightarrow A = \int_M dM = c\int_0^{2\pi}\int_{-a}^{a}\cosh^2 z/c\,dz\,d\theta$$

$$= 2\pi c\int_{-a}^{a}\frac{\cosh 2z/c + 1}{2}dz = 2\pi c\left(\frac{\sinh 2z/c}{4} + \frac{z}{2}\right)\Big|_{-a}^{a} \longmapsto \infty$$

Helicoid

$\vec{x} = r\hat{r} + b\theta\hat{z}$

$\Rightarrow ds^2 = dr^2 + (r^2+b^2)\,d\theta^2$

$\Rightarrow dM = \sqrt{r^2+b^2}\,dr \wedge d\theta$

$\Rightarrow A = \int_H dM = \int_0^{2\pi}\int_0^a \sqrt{r^2+b^2}\,dr\,d\theta$

$> 2\pi \int_0^a r\,dr\,d\theta = \pi a^2 \to \infty$

2π for just 1 turn — also → ∞

Torus

$\vec{x} = (R + \rho\cos\psi)\,\hat{r} + \rho\sin\psi\,\hat{z}$

$\Rightarrow ds^2 = (R+\rho\cos\psi)^2 d\theta^2 + \rho^2 d\psi^2$

$\Rightarrow dM = \rho(R+\rho\cos\psi)\,d\theta \wedge d\psi$

$\Rightarrow A = \int_T dM = \int_0^{2\pi}\int_0^{2\pi} \rho(R+\rho\cos\psi)\,d\theta\,d\psi$

$= 2\pi\rho\,(R\psi - \cancel{\rho\sin\psi})\Big|_0^{2\pi}$

$= 4\pi^2\rho R$