REVIEW OF DIFFERENTIAL FORMS

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Abstract

These notes are intended as a *very* concise summary of the basic idea of differential forms and curvature, intended for those studying general relativity. Please bear in mind when reading this document that it is not necessary to follow the details of each and every step. Rather, it is important to have a basic grasp of what is going on, and to be able to use the formalism here to calculate curvature.

1 Introduction

This is a *very* concise summary of the basic idea of curvature, intended for those studying general relativity. Along the way, we encounter covariant differentiation and affine connections in considerable generality, and then discusses how to compute curvature in two important special cases, namely using a coordinate basis and using an orthonormal basis. The latter method requires familiarity with differential forms; the former does not.

Please bear in mind when reading this document that it is not necessary to follow the details of each and every step. Rather, it is important to have a basic grasp of what is going on, and to be able to calculate curvature using any *one* method.

2 Motivation: Differential Forms in $\mathbb{R}^3$

What are the possible integrands in (Euclidean) $\mathbb{R}^3$? *Line integrals* have integrands of the form $\vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$, *surface integrals* have integrands of the form $\vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{A}}$, and volume integrals have integrands of the form $f\,dV$. These objects are our prototypes for *1-forms*, *2-forms*, and *3-forms*, respectively. We also include functions f as *0-forms*; "integration" of a 0-form consists simply of evaluating the function at given points.

2.1 1-forms

Consider the *Master Formula*

$$df = \vec{\nabla} f \cdot d\vec{\boldsymbol{r}} \tag{1}$$

relating the multivariable differential df to the gradient. Here, $d\vec{\boldsymbol{r}}$ is the infinitesimal displacement vector between nearby points, given in rectangular coordinates [1]

$$d\vec{\boldsymbol{r}} = dx\,\hat{\boldsymbol{x}} + dy\,\hat{\boldsymbol{y}} + dz\,\hat{\boldsymbol{z}} \tag{2}$$

More generally, given any vector field $\vec{\boldsymbol{F}}$, the *1-form* associated with $\vec{\boldsymbol{F}}$ is given by

$$F = \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} \tag{3}$$

We write $\bigwedge^1(\mathbb{R}^3)$ for the collection of 1-forms on (Euclidean) $\mathbb{R}^3$; when the underlying space is clear from context, we often abbreviate this to $\bigwedge^1$. Since

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz \tag{4}$$

it is clear that $\{dx, dy, dz\}$ is a *basis* for $\bigwedge^1$, just as $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$ is a basis for vector fields. [2] In fact, basis transformations naturally involve Jacobians, since (4) naturally involves both calculus and linear algebra.

Furthermore, there is a natural inner product on 1-forms, given by

$$g(\vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}, \vec{\boldsymbol{G}} \cdot d\vec{\boldsymbol{r}}) = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{G}} \tag{5}$$

This inner product makes $\{dx, dy, dz\}$ into an *orthonormal* basis for $\bigwedge^1$. This fact is often expressed by writing

$$ds^2 = d\vec{\boldsymbol{r}} \cdot d\vec{\boldsymbol{r}} = dx^2 + dy^2 + dz^2 \tag{6}$$

and noting that the infinitesimal distances in this statement of the Pythagorean Theorem are precisely our orthonormal basis vectors. It is well worth considering what the analogous statements would be in other coordinate systems, such as spherical coordinates.

2.2 Multiplication of 1-forms

We would like to continue in the same vein, and define 2-forms to be objects of the form $\vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{A}}$. However, this isn't quite right. The problem is that, as a vector object, $d\vec{\boldsymbol{A}}$ has an orientation. Put differently, we want $dy\,dx$ to be different from $dx\,dy$; there should be a relative factor of -1 between these two expressions.

We therefore *define* an antisymmetric product on 1-forms, so that

$$dy \wedge dx = -dx \wedge dy \tag{7}$$

[1] More formally, $d\vec{\boldsymbol{r}}$ is a *vector-valued 1-form*.

[2] These spaces are *modules* over the ring of functions, which just means that they behave like vector spaces, but with scalars that are *functions*, rather than numbers.

which is called the *wedge* or *exterior* product. We henceforth interpret objects such as $\vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{A}}$ as involving linear combinations of such products; these are *2-forms*.

We extend the wedge product to higher rank differential forms by associativity. In $\mathbb{R}^3$, we have the following spaces and bases:

$$\bigwedge^0 : \quad \{1\} \tag{8}$$

$$\bigwedge^1 : \quad \{dx, dy, dz\} \tag{9}$$

$$\bigwedge^2 : \quad \{dy \wedge dz, dz \wedge dx, dx \wedge dy\} \tag{10}$$

$$\bigwedge^3 : \quad \{dx \wedge dy \wedge dz\} \tag{11}$$

Note that all higher rank forms are 0, since $dx \wedge dx = 0$, etc.

2.3 Duality

There is clearly a relationship between 0-forms and 3-forms, given by relating f and $f\,dV$. We can therefore define a map from 0-forms to 3-forms, which we write as

$$*f = f\,dV \tag{12}$$

Similarly, relating $\vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{A}}$ yields a relationship between 1-forms and 2-forms, which we write as

$$*(\vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{r}}) = \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{A}} \tag{13}$$

In three (Euclidean) dimensions, we also write $*$ for the inverse maps, so that

$$*(f\,dV) = f \tag{14}$$

and

$$*(\vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{A}}) = \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{r}} \tag{15}$$

We can work out the action of the *Hodge dual* map $*$ on a basis at each rank. The standard basis of 1-forms in $\mathbb{R}^3$ is the set $\{dx, dy, dz\}$; any 1-form can be expressed as a linear combination of these basis 1-forms, with coefficients that are *functions*. Similarly, the standard basis of 2-forms is $\{dy \wedge dz, dz \wedge dx, dx \wedge dy\}$, and the standard basis of 3-forms is $\{dx \wedge dy \wedge dz\}$. Finally, the standard basis of 0-forms is just $\{1\}$, as any function is a multiple of the constant function 1 (with a coefficient that is a function). The action of $*$ is therefore given by

$$*1 = dx \wedge dy \wedge dz \tag{16}$$

$$*dx = dy \wedge dz \tag{17}$$

$$*dy = dz \wedge dx \tag{18}$$

$$*dz = dx \wedge dy \tag{19}$$

$$
\begin{aligned}
*(dy \wedge dz) &= dx & (20)\\
*(dz \wedge dx) &= dy & (21)\\
*(dx \wedge dy) &= dz & (22)\\
*(dx \wedge dy \wedge dz) &= 1 & (23)
\end{aligned}
$$

along with the linearity property

$$
(f\,\alpha + \beta) = f\,{\alpha} + {*\beta} \tag{24}
$$

for any function f and any p-forms α, β.

2.4 Derivatives

How do we differentiate differential forms?

It's pretty clear how to differentiate 0-forms (functions): Take the gradient. The 1-form associated with $\vec{\nabla} f$ is given by the Master Formula (1).

Our remaining vector derivative operators are the curl, which takes vector fields to vector fields, and the divergence, which takes vector fields to scalar fields. Since adding a "d" takes a p-form to a $(p+1)$-form, it is natural to *define* differentiation of 1- and 2-forms by

$$
\begin{aligned}
d\left(\vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}\right) &= \left(\vec{\nabla} \times \vec{\boldsymbol{F}}\right) \cdot d\vec{\boldsymbol{A}} & (25)\\
d\left(\vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{A}}\right) &= \left(\vec{\nabla} \cdot \vec{\boldsymbol{F}}\right) dV & (26)
\end{aligned}
$$

Using the relationship between vector fields $\vec{\boldsymbol{G}}$ and 1-forms $\vec{\boldsymbol{G}} \cdot d\vec{\boldsymbol{r}}$, we see that $*dF$ is the 1-form associated with $\vec{\nabla} \times \vec{\boldsymbol{F}}$, and $*d{*F}$ is the 0-form $\vec{\nabla} \cdot \vec{\boldsymbol{F}}$. We can regard these relations as the definitions of the curl and divergence of a 1-form

$$
\begin{aligned}
*dF &= \mathrm{curl}(F) & (27)\\
*d{*F} &= \mathrm{div}(F) & (28)
\end{aligned}
$$

and of course we have

$$
df = \mathrm{grad}(f) \tag{29}
$$

for 0-forms. [3]

[3]In casual usage, one often refers to 1-forms as "vectors", leading one to interpret F and df (and $\mathrm{curl}(F)$) as "vectors" when they are really 1-forms. More precisely, if $F = \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$, then we have

$$
\begin{aligned}
\mathrm{curl}(F) &= \left(\vec{\nabla} \times \vec{\boldsymbol{F}}\right) \cdot d\vec{\boldsymbol{r}} & (30)\\
\mathrm{div}(F) &= \vec{\nabla} \cdot \vec{\boldsymbol{F}} & (31)\\
\mathrm{grad}(f) &= \vec{\nabla} f \cdot d\vec{\boldsymbol{r}} & (32)
\end{aligned}
$$

Why have we rewritten vector calculus in this new language? Because it unifies the results. By convention, d acting on a 3-form is zero; the result should be a 4-form, but there are no (nonzero) 4-forms in three dimensions. We therefore have the remarkable identity

$$d^2 = 0 \tag{33}$$

when acting on p-forms for any p. Furthermore, all of the integral theorems of vector calculus can be combined into the single statement

$$\int_D d\alpha = \int_{\partial D} \alpha \tag{34}$$

where D is a p-dimensional region in $\mathbb{R}^3$, α is a $(p-1)$-form, and ∂D denotes the boundary of D.

3 Differential Forms

In general, $d\vec{\boldsymbol{r}}$ must be determined from the geometry, or given explicitly. We emphasize that we do *not* assume that the dot product is positive definite; there may be nonzero vectors such that $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}} \le 0$. A vector will be considered to be normalized if

$$|\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}| = 1 \tag{35}$$

We can nonetheless regard (1) as the *definition* of the gradient, and (3) as the definition of 1-forms. Wedge products carry through as before. An important property of the wedge product is that

$$\beta \wedge \alpha = (-1)^{pq} \alpha \wedge \beta \tag{36}$$

where $\alpha \in \bigwedge^p$ and $\beta \in \bigwedge^q$.

In order to generalize the notion of duality, we must specify one further piece of information, the *orientation* ω, also called the *volume element*, that gives us the natural basis for n-forms, that is, which sets the scale and orientation for (n-dimensional) volume. For example, in $\mathbb{R}^3$, we have

$$\omega = dx \wedge dy \wedge dz \tag{37}$$

This construction works in general: If $\{\sigma^i\}$ is an orthonormal basis of $\bigwedge^1$, then we assume

$$\omega = \sigma^1 \wedge ... \wedge \sigma^n \tag{38}$$

unless explicitly stated otherwise.

The Hodge dual operation can now be defined implicitly by

$$\alpha \wedge *\beta = g(\alpha, \beta)\,\omega \tag{39}$$

where $\alpha, \beta \in \bigwedge^p$, and where g is extended to higher-rank forms by

$$g(\alpha^1 \wedge ... \wedge \alpha^p, \beta^1 \wedge ... \wedge \beta^p) = \left| g(\alpha^i, \beta^j) \right| \tag{40}$$

where the last expression denotes the determinant of the matrix whose generic element is given, and where $\alpha^i, \beta^j \in \bigwedge^1$.

In practice, it is much easier to apply (39) to an orthonormal basis element, since

$$*(\sigma^1 \wedge ... \wedge \sigma^p) = g(\sigma^1, \sigma^1)...g(\sigma^p, \sigma^p)\, \sigma^{p+1} \wedge ... \wedge \sigma^n \tag{41}$$

and each of the factors $g(\sigma^i, \sigma^i)$ is ± 1. However, be careful when using (41): It only holds in an orthonormal basis, and you must still consider permutation factors when computing, say, $*(\sigma^2 \wedge \sigma^5)$.

Some simple consequences of these formulas are:

$$*1 = \omega \tag{42}$$

$$** = (-1)^{p(n-p)+s} \tag{43}$$

where s is the *signature* of the inner product g, namely the number of (orthonormal) basis elements whose squared magnitude is -1.

What about differentiation? Again, differentiating 0-forms is easy; just take f to df. We generalize this operation to p-forms by defining

$$d(f\, dx^1 \wedge ... \wedge dx^p) = df \wedge dx^1 \wedge ... \wedge dx^p \tag{44}$$

and extending by linearity. It is important to realize that the expressions in the wedge product are *coordinate* 1-forms, and typically *not* elements of the orthonormal basis.

Important properties of this process of *exterior differentiation* that follow from this definition are

$$d^2 = 0 \tag{45}$$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \tag{46}$$

where $\alpha \in \bigwedge^p$.

4 Connections

Suppose we are given the infinitesimal displacement in the form

$$d\vec{\boldsymbol{r}} = \sigma^i \hat{\boldsymbol{e}}_i \tag{47}$$

where $\{\hat{\boldsymbol{e}}_i\}$ is an orthonormal vector basis and the σ^i are 1-forms; $d\vec{\boldsymbol{r}}$ is thus a *vector-valued 1-form*, and, via (3), provides a map from vector to 1-forms. We think of v and $\vec{\boldsymbol{v}}$ as physically equivalent, so we impose the condition

$$|v| = |\vec{\boldsymbol{v}}| \tag{48}$$

which leads by polarization to the inner product on 1-forms given by (5). Not surprisingly, $\{\sigma^i\}$ is an orthonormal 1-form basis under this inner product. Furthermore, the inverse isomorphism from 1-forms to vectors is given by

$$\vec{\boldsymbol{v}} = g(v, d\vec{\boldsymbol{r}}) \tag{49}$$

Finally, we note that $\{\sigma^i\}$ is just the dual basis to $\{\hat{\boldsymbol{e}}_i\}$, under the action

$$v(\vec{\boldsymbol{w}}) := \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} \tag{50}$$

with v and $\vec{\boldsymbol{v}}$ related as above.

We wish to extend the exterior derivative operator on differential forms to (ordinary) vectors. We define

$$\omega_{ij} := \hat{\boldsymbol{e}}_i \cdot d\hat{\boldsymbol{e}}_j \tag{51}$$

and the goal is now to determine the 1-forms ω_{ij}. We first impose the condition that d be *metric compatible*, that is, that

$$d(\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}) = d\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} + \vec{\boldsymbol{v}} \cdot d\vec{\boldsymbol{w}} \tag{52}$$

from which it follows immediately that

$$0 = d(\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j) = d\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j + d\hat{\boldsymbol{e}}_j \cdot \hat{\boldsymbol{e}}_i \tag{53}$$

or in other words

$$\omega_{ji} + \omega_{ij} = 0 \tag{54}$$

We also require that d be *torsion free*, which means by definition that [4]

$$d(d\vec{\boldsymbol{r}}) = 0 \tag{55}$$

Working out this condition, we have [5]

$$d(d\vec{\boldsymbol{r}}) = d\sigma^i \hat{\boldsymbol{e}}_i - \sigma^i \wedge d\hat{\boldsymbol{e}}_i = d\sigma^i \hat{\boldsymbol{e}}_i - \sigma^k \wedge d\hat{\boldsymbol{e}}_k \tag{56}$$

so that

$$d(d\vec{\boldsymbol{r}}) \cdot \hat{\boldsymbol{e}}_j = d\sigma^i \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j - \sigma^k \wedge \omega_{jk} \tag{57}$$

Noting that $g_{ij} = \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j$ is constant (and in fact 0 or ± 1), we define the *connection 1-forms* ${\omega^i}_j$ by requiring

$$\omega_{jk} = g_{ji}\, {\omega^i}_k \tag{58}$$

and the torsion-free condition becomes the (first) Cartan structure equation

$$d\sigma^i + {\omega^i}_k \wedge \sigma^k = 0 \tag{59}$$

It turns out that there is a unique torsion-free, metric-compatible connection, which is known as the *Levi-Civita connection*. In other words, the connection 1-forms ${\omega^i}_j$ are uniquely determined by (54) and (59). Although a formula can be given for the connection, the most efficient way to determine the connection is usually to guess a solution of these equations. If it works, you're done!

[4] This condition does *not* automatically follow from the requirement that $d^2 = 0$ on differential forms, since it is not obvious that $d\vec{\boldsymbol{r}}$ is d of anything; the position vector $\vec{\boldsymbol{r}}$ may not be in our vector space.

[5] We adopt the *Einstein summation convention*; repeated indices are summed over.

5 Curvature

The *curvature 2-forms* $\Omega^i{}_j$ are defined by the (second) Cartan structure equation

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_m \wedge \omega^m{}_j \tag{60}$$

Expanding with respect to our basis, we can write

$$\omega^k{}_i = \Gamma^k{}_{ij}\,\sigma^j \tag{61}$$

$$\Omega^i{}_j = \frac{1}{2} R^i{}_{jkl}\,\sigma^k \wedge \sigma^l \tag{62}$$

which defines the quantities on the right-hand-side. The latter expression corresponds to a tensor called the *Riemann curvature tensor*, whose components are $R^i{}_{jkl}$. The "connection components" $\Gamma^k{}_{ij}$ are called *Christoffel symbols*, and are *not* the components of a tensor. (In particular, they can all vanish in one basis but not in another, which is not possible for tensor components.)

Two contractions of the Riemann tensor are important in relativity. These are the *Ricci tensor*, whose components are defined by $R_{ij} = R^m{}_{imj}$, and the *Ricci scalar*, which is the "trace" of the Ricci tensor, defined by $R = g^{ij} R_{ij}$, where g^{ij} denotes the (components of the) inverse of the metric tensor, whose components are g_{ij}.