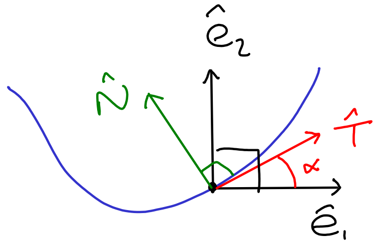
Geodesic Curvature



$$\hat{T} = \cos\alpha\,\hat{e}_1 + \sin\alpha\,\hat{e}_2$$

$$\Rightarrow d\hat{T} = \overbrace{\left(-\sin\alpha\,\hat{e}_1 + \cos\alpha\,\hat{e}_2\right)}^{\hat{N}} d\alpha + \cos\alpha\,\omega^2{}_1\,\hat{e}_2 + \sin\alpha\,\omega^1{}_2\,\hat{e}_1$$

$$= \hat{N}\,\underbrace{\left(d\alpha - \omega^1{}_2\right)}_{\kappa_g}$$

Note: $\hat{N} = \hat{e}_3 \times \hat{T}$

geodesic curvature:

$$\boxed{\kappa_g\,ds = d\hat{T}\cdot\hat{N} = d\alpha - \omega^1{}_2}$$

geodesic: $\kappa_g = 0$

Ex: Sphere

line of latitude; $\hat{T} = \hat{\phi}$

(why?) $\hat{N} = \hat{r} \times \hat{\phi} = -\hat{\theta}$

$$\therefore \kappa_g\,ds = d\hat{\phi}\cdot(-\hat{\theta}) = \cos\theta\,d\phi$$

$$\therefore \kappa_g = 0 \iff \theta = \frac{\pi}{2}$$

Great circles!

Geodesic Triangles

What is the total curvature around a closed curve in some surface?

Integrate!

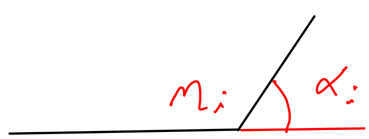
$$\oint_c \kappa_g \, ds = \oint_c d\alpha - \oint_c \omega^1{}_2$$

$$= 2\pi - \int_{\substack{\text{interior}\\ \text{of } C}} d\omega^1{}_2 \qquad \text{(Stokes' Thm!)}$$

$$= 2\pi - \int K \omega$$

Gauß curvature

$$\Rightarrow \boxed{\int K\omega + \oint \kappa_g \, ds = 2\pi}$$

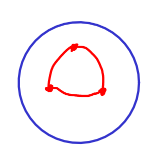


If the curve has corners, e.g. a polygon

$$\boxed{\begin{aligned} \int K\omega + \oint \kappa_g \, ds &= 2\pi - \sum \epsilon_i \\ &= 2\pi - \sum (\pi - m_i) \end{aligned}}$$

Example: geodesic triangle on sphere

$$\Rightarrow \frac{1}{r^2}(\text{area}) + 0 = 2\pi - 3\pi + \text{angle sum}$$



$$\Rightarrow \boxed{\frac{1}{r^2}(\text{area}) = \text{angle sum} - \pi}$$

$\Rightarrow$ no similar triangles!

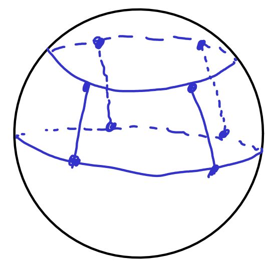
In plane: $K = 0 \Rightarrow$ angle sum $= \pi$

Gauß - Bonnet Theorem

Consider (nongeodesic) rectangles

$$\Longrightarrow \quad \int K\omega + \oint K_g\, ds = 2\pi - 4\pi + \sum m_i$$

Now decompose compact surface into rectangles



① $\sum \oint K_g\, ds = 0$

since each edge traversed both ways

② $\sum (2\pi - 4\pi) = -2\pi f$

③ $\sum \left(\sum m_i\right) = 2\pi v$

$f = 6$
$e = 12$ $\Rightarrow$ $\chi = 8 - 12 + 6 = 2$
$v = 8$

$$\therefore \int_S K\omega = 2\pi v - 2\pi f$$

for rectangles

④ each face has 4 edges — but each edge is on 2 faces

$\Rightarrow e = 4f/2 = 2f \Rightarrow -f = f - e$

$\Rightarrow v - f = v - e + f = \chi$

Euler characteristic (independent of decomposition)

$$\therefore \boxed{\int_S K\omega = 2\pi \chi(S)}$$

geometry　　topology

Fact: $\chi(\text{torus}) = 0 \Rightarrow K_{\text{torus}}$ can not be everywhere positive