Bianchi Identities

$$0 = d^2\vartheta^i = d(d\vartheta^i) = d(-\omega^i{}_k \wedge \vartheta^k)$$
$$= -d\omega^i{}_k \wedge \vartheta^k + \omega^i{}_k \wedge d\vartheta^k$$
$$= -d\omega^i{}_j \wedge \vartheta^j + \omega^i{}_k \wedge (-\omega^k{}_j \wedge \vartheta^j)$$
$$= -(d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j) \wedge \vartheta^j$$
$$= -\Omega^i{}_j \wedge \vartheta^j$$

$$\therefore \boxed{\Omega^i{}_j \wedge \vartheta^j = 0}$$

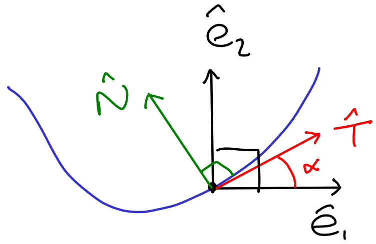
1st Bianchi id

$$0 = d^2\omega^i{}_j = d(d\omega^i{}_j)$$
$$= d(\Omega^i{}_j - \omega^i{}_k \wedge \omega^k{}_j)$$
$$= d\Omega^i{}_j - d\omega^i{}_k \wedge \omega^k{}_j + \omega^i{}_k \wedge d\omega^k{}_j$$
$$= d\Omega^i{}_j - (\Omega^i{}_k - \cancel{\omega^i{}_m \wedge \omega^m{}_k}) \wedge \omega^k{}_j$$
$$+ \omega^i{}_k \wedge (\Omega^k{}_j - \cancel{\omega^k{}_m \wedge \omega^m{}_j})$$

$$\therefore \boxed{d\Omega^i{}_j + \omega^i{}_k \wedge \Omega^k{}_j - \Omega^i{}_k \wedge \omega^k{}_j = 0}$$

2nd Bianchi id

Geodesic Curvature



$$\hat{T} = \cos\alpha\, \hat{e}_1 + \sin\alpha\, \hat{e}_2$$

$$\Rightarrow d\hat{T} = \big(\overbrace{-\sin\alpha\, \hat{e}_1 + \cos\alpha\, \hat{e}_2}^{\hat{N}}\big)\, d\alpha + \cos\alpha\, \omega^2{}_1\, \hat{e}_2 + \sin\alpha\, \omega^1{}_2\, \hat{e}_1$$

$$= \hat{N}\, \underbrace{(d\alpha - \omega^1{}_2)}_{K_g}$$

Note: $\hat{N} = \hat{e}_3 \times \hat{T}$

geodesic curvature:

$$\boxed{K_g\, ds = d\hat{T} \cdot \hat{N} = d\alpha - \omega^1{}_2}$$

geodesic: $K_g = 0$

Ex: Sphere

line of latitude: $\hat{T} = \hat{\Phi}$

(why?) $\hat{N} = \hat{r} \times \hat{\Phi} = -\hat{\theta}$

$$\therefore K_g\, ds = d\hat{\Phi} \cdot (-\hat{\theta}) = \cos\theta\, d\Phi$$

$$\therefore K_g = 0 \iff \theta = \frac{\pi}{2}$$

Great circles!