MTH 434 — HW #1

(Solution)

Let $\vec{\boldsymbol{u}} = u_x\hat{\boldsymbol{x}} + u_y\hat{\boldsymbol{y}} + u_z\hat{\boldsymbol{z}}$*. Find two vectors* $\vec{\boldsymbol{v}}$ *and* $\vec{\boldsymbol{w}}$ *such that* $\vec{\boldsymbol{u}} = \vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}}$*.*

Idea:

If $\vec{\boldsymbol{u}} = \vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}}$, then $\vec{\boldsymbol{u}} \perp \vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{u}} \perp \vec{\boldsymbol{w}}$. Furthermore, if $\vec{\boldsymbol{v}} \parallel \vec{\boldsymbol{w}}$, then $\vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}} = 0$, so that (of course) $\{\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}\}$ must be linearly independent. This suggests a procedure: Find *any* linearly independent vectors $\vec{\boldsymbol{v}}$, $\vec{\boldsymbol{w}}$ which are both perpendicular to $\vec{\boldsymbol{u}}$. Then $\vec{\boldsymbol{u}} \parallel \vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}}$, and it only remains to suitably rescale $\vec{\boldsymbol{v}}$ and/or $\vec{\boldsymbol{w}}$.

Solution:

Suppose $\vec{\boldsymbol{u}} = u_x\hat{\boldsymbol{x}} + u_y\hat{\boldsymbol{y}} + u_z\hat{\boldsymbol{z}}$.

Consider first the case $u_x = 0$, which implies that $\vec{\boldsymbol{u}} \perp \hat{\boldsymbol{x}}$. The properties of the cross product imply that $\hat{\boldsymbol{x}} \times \vec{\boldsymbol{u}}$ is perpendicular to both $\hat{\boldsymbol{x}}$ and $\vec{\boldsymbol{u}}$. Thus, $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{x}} \times \vec{\boldsymbol{u}} = -u_z\,\hat{\boldsymbol{y}} + u_y\,\hat{\boldsymbol{z}}$ are two linearly independent vectors perpendicular to $\vec{\boldsymbol{u}}$, and direct computation shows that $(\hat{\boldsymbol{x}} \times \vec{\boldsymbol{u}}) \times \hat{\boldsymbol{x}} = \vec{\boldsymbol{u}}$.

Now suppose that $u_x \neq 0$, so that $\vec{\boldsymbol{u}}$ has a nonzero $\hat{\boldsymbol{x}}$ component. An easy way to obtain two linearly independent vectors orthogonal to $\vec{\boldsymbol{u}}$ is to assume that one is in the xy-plane (no $\hat{\boldsymbol{z}}$ component) and the other is in the xz-plane (no $\hat{\boldsymbol{y}}$ component). Two such vectors are $u_x\hat{\boldsymbol{y}} - u_y\hat{\boldsymbol{x}}$ and $u_z\hat{\boldsymbol{x}} - u_x\hat{\boldsymbol{z}}$; notice that these vectors are just $\hat{\boldsymbol{z}} \times \vec{\boldsymbol{u}}$ and $\hat{\boldsymbol{y}} \times \vec{\boldsymbol{u}}$, respectively. Rescaling these vectors slightly and taking the cross product, we obtain

$$\vec{\boldsymbol{u}} = u_x \left(\hat{\boldsymbol{y}} - \frac{u_y}{u_x}\hat{\boldsymbol{x}}\right) \times \left(\hat{\boldsymbol{z}} - \frac{u_z}{u_x}\hat{\boldsymbol{x}}\right)$$

from which various choices of $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ can be read off, depending on the scaling. In this form, it is obvious that $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ are independent (and that we must require $u_x \neq 0$).

(We have also verified a special case of the identify $(\vec{\boldsymbol{a}} \times \vec{\boldsymbol{u}}) \times (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{u}}) = ((\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) \cdot \vec{\boldsymbol{u}})\,\vec{\boldsymbol{u}}$.)

This is not the only solution! (Note that $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ are **not** perpendicular. This condition can of course also be satisfied, but the solution becomes more complicated.)

Alternate Solution:

Rewrite each basis vector as a cross product of the other two basis vectors. The goal is to factor the resulting "polynomial" $\vec{\boldsymbol{u}} = u_x\,(\hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}}) + u_y\,(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{x}}) + u_z\,(\hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}})$.

If $u_x = 0$, then $\vec{\boldsymbol{u}} = u_y\,(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{x}}) + u_z\,(\hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}})$, which has a common factor of $\hat{\boldsymbol{x}}$. Thus, $\vec{\boldsymbol{u}} = (u_y\hat{\boldsymbol{z}} - u_z\hat{\boldsymbol{y}}) \times \hat{\boldsymbol{x}}$, which is the same solution as above.

If $u_x \neq 0$, we can still factor the first two terms as $u_x\,(\hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}}) + u_y\,(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{x}}) = (u_x\hat{\boldsymbol{y}} - u_y\hat{\boldsymbol{x}}) \times \hat{\boldsymbol{z}}$. We'd like to include the third term, $u_z\,(\hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}})$. Reversing the order and assuming $u_x \neq 0$, we get

$$u_z\,(\hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}}) = -u_z\,(\hat{\boldsymbol{y}} \times \hat{\boldsymbol{x}}) = \hat{\boldsymbol{y}} \times (-u_z\hat{\boldsymbol{x}}) = u_x\hat{\boldsymbol{y}} \times \left(-\frac{u_z}{u_x}\right)\hat{\boldsymbol{x}}$$

Now for the tricky part: The first two terms have the factor $u_x\hat{\boldsymbol{y}} - u_y\hat{\boldsymbol{x}}$, not $u_x\hat{\boldsymbol{y}}$ alone. But if we use the first factor in the third term, the term involving u_y disappears (since $\hat{\boldsymbol{x}} \times \hat{\boldsymbol{x}} = \vec{\boldsymbol{0}}$)! In other words,

$$u_z\,(\hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}}) = \ldots = u_x\hat{\boldsymbol{y}} \times \left(-\frac{u_z}{u_x}\right)\hat{\boldsymbol{x}} = (u_x\hat{\boldsymbol{y}} - u_y\hat{\boldsymbol{x}}) \times \left(-\frac{u_z}{u_x}\right)\hat{\boldsymbol{x}}$$

Putting this all together, we get

$$\vec{\boldsymbol{u}} = (u_x\hat{\boldsymbol{y}} - u_y\hat{\boldsymbol{x}}) \times \left(\hat{\boldsymbol{z}} - \frac{u_z}{u_x}\hat{\boldsymbol{x}}\right)$$

which again agrees with the solution above. **This is not the only solution!**