Higher Rank Forms

Have: $g(dx, dx) = 1$

Want: $g(dx \wedge dy, dx \wedge dy) = 1$

If true, g linear $\Rightarrow$

$$g(\alpha \wedge \beta, dx \wedge dy) = \alpha_x \beta_y - \alpha_y \beta_x$$

$$= \begin{vmatrix} g(\alpha, dx) & g(\alpha, dy) \\ g(\beta, dx) & g(\beta, dy) \end{vmatrix}$$

$\therefore$ define $\boxed{g(\alpha \wedge \beta, \gamma \wedge \delta) = \begin{vmatrix} g(\alpha, \gamma) & g(\alpha, \delta) \\ g(\beta, \gamma) & g(\beta, \delta) \end{vmatrix}}$

for $\alpha, \beta, \gamma, \delta \in \Lambda^1$

$$\Rightarrow g(\alpha^1 \wedge \dots \wedge \alpha^p, \beta^1 \wedge \dots \wedge \beta^p) = \left| g(\alpha^i, \beta^j) \right|$$

Orthonormal basis:

$$g(\sigma^i, \sigma^j) = \pm \delta^{ij}$$

$$\Rightarrow \quad g(\sigma^I, \sigma^J) = \pm \delta^{IJ}$$

Special case: $g(1, 1) = 1$

- linear ✓
- symmetric: $|A^T| = |A|$
- non-degenerate: $\{\sigma^I\}$ orthonormal

Schwarz Inequality

Work in $\mathbb{R}^2$. $\alpha, \beta \in \Lambda^1 \Rightarrow$

$$g(\alpha\wedge\beta, \alpha\wedge\beta) = \begin{vmatrix} g(\alpha,\alpha) & g(\alpha,\beta) \\ g(\beta,\alpha) & g(\beta,\beta) \end{vmatrix}$$

$$= g(\alpha,\alpha)\,g(\beta,\beta) - g(\alpha,\beta)^2$$

But $\alpha\wedge\beta = f\,dx\wedge dy$ for some f

$$\Rightarrow g(\alpha\wedge\beta, \alpha\wedge\beta) = f^2 g(dx\wedge dy, dx\wedge dy)$$
$$= f^2 g(dx, dx)\, g(dy, dy)$$
$$= f^2 \geq 0$$

$$\therefore \boxed{g(\alpha,\beta)^2 \leq g(\alpha,\alpha)\, g(\beta,\beta)}$$

Schwarz inequality

Work in $\mathbb{M}^2$.

$$\alpha\wedge\beta = f\,dx\wedge dt$$
$$\Rightarrow g(\alpha\wedge\beta, \alpha\wedge\beta) = f^2 g(dx\wedge dt, dx\wedge dt)$$
$$= f^2 g(dx, dx)\, g(dt, dt)$$
$$= -f^2 \leq 0$$

$$\therefore \boxed{g(\alpha,\beta)^2 \geq g(\alpha,\alpha)\, g(\beta,\beta)}$$

reverse Schwarz inequality

Can generalize to $n > 2$

since α, β span a 2-d subspace

In the + def case:

$$(\alpha+\beta)\cdot(\alpha+\beta)$$
$$= \alpha\cdot\alpha + 2\alpha\cdot\beta + \beta\cdot\beta$$
$$\leq \alpha\cdot\alpha + 2\sqrt{\alpha\cdot\alpha}\sqrt{\beta\cdot\beta} + \beta\cdot\beta$$
$$= \left(\sqrt{\alpha\cdot\alpha} + \sqrt{\beta\cdot\beta}\right)^2$$

$$\therefore \boxed{|\alpha+\beta| \leq |\alpha| + |\beta|}$$

triangle inequality

"shortest distance between 2 pts is along a straight line"

SR: longest time between 2 events is along a straight line!

twin paradox!

exercise keeps you young...

Orientation

Recall: $|\Lambda^n| = 1$

$\therefore \exists$ exactly 2 elements $\omega \in \Lambda^n$ with (squared) norm 1, that is, satisfying

$$\boxed{g(\omega,\omega) = 1}$$

An orientation $\omega \in \Lambda^n$ is a choice of one of these elements.

(The only other choice is $-\omega$)

Ex: $\mathbb{R}^2$: $\omega = dx \wedge dy$
$= r\,dr \wedge d\theta$

$\mathbb{M}^2$: $\omega = dx \wedge dt$

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basis independent!