Properties of d

What is $d(df)$?

$$df = \frac{\partial f}{\partial x^i}\, dx^i$$

$$d(df) = d\left(\frac{\partial f}{\partial x^i}\right) dx^i$$

$$= \frac{\partial^2 f}{\partial x^j \partial x^i}\, dx^j \wedge dx^i = 0 \quad !$$

sym antisym

$$\sum_{i,j} = \sum_{i<j} + \sum_{i>j} = \sum_{i<j} - \sum_{j>i} = 0$$

Same argument works for $d(d\alpha)$:

$$d(f\, dx^I) = df \wedge dx^I = \frac{\partial f}{\partial x^i}\, dx^i \wedge dx^I$$

$$d(\qquad) = \frac{\partial^2 f}{\partial x^j \partial x^i}\, dx^j \wedge dx^i \wedge dx^I$$

$$\boxed{d^2 = 0}$$

$$\vec{\nabla} \times \vec{\nabla} f \longleftrightarrow * d(df) = 0$$

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} \longleftrightarrow * d * (* df) = * ddf = 0$$

Product Rules

"$d(fg) = f\,dg + g\,df$"

$\alpha \in \Lambda^p,\ \beta \in \Lambda^q$

what is $d(\alpha \wedge \beta)$?

$$\begin{aligned}
d(f\,dx^I \wedge g\,dx^J) &= d(fg\,dx^I \wedge dx^J) \\
&= d(fg) \wedge dx^I \wedge dx^J \\
&= (g\,df + f\,dg) \wedge dx^I \wedge dx^J \\
&= df \wedge dx^I \wedge g\,dx^J + f\,dg \wedge dx^I \wedge dx^J \\
&= df \wedge dx^I \wedge g\,dx^J + (-1)^p f\,dx^I \wedge dg \wedge dx^J \\
&= d(f\,dx^I) \wedge g\,dx^J + (-1)^p f\,dx^I \wedge d(g\,dx^J)
\end{aligned}$$

$$\therefore \boxed{d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta}$$

$$\begin{aligned}
\vec{\nabla}(fg) &= (\vec{\nabla} f) g + f \vec{\nabla} g \\
\vec{\nabla} \times (f\vec{G}) &= \vec{\nabla} f \times \vec{G} + f\, \vec{\nabla} \times \vec{G} \\
\vec{\nabla} \cdot (f\vec{G}) &= \vec{\nabla} f \cdot \vec{G} + f\, \vec{\nabla} \cdot \vec{G} \\
\vec{\nabla} \cdot (\vec{F} \times \vec{G}) &= (\vec{\nabla} \times \vec{F}) \cdot \vec{G} - \vec{F} \cdot (\vec{\nabla} \times \vec{G})
\end{aligned}$$

$$\begin{aligned}
d(fg) &= (df)g + f\,dg \\
*d(fG) &= *(df \wedge G) + *(f\,dG) \\
d(fG) &= *d(f *G) \\
&= *(df \wedge *G) + *(f\,d*G) \\
*d**(F \wedge G) &= *d(F \wedge G) \\
&= *(dF \wedge G) - *(F \wedge dG) \\
&= *(**dF \wedge G) - *(F \wedge **dG)
\end{aligned}$$

Uniqueness

Thm: $\exists!\, d : \Lambda^p \to \Lambda^{p+1}$:

① linear: $d(a\alpha + \beta) = a\, d\alpha + d\beta$

② product rule: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$

③ $d^2 \alpha = 0$

④ $df = \dfrac{\partial f}{\partial x^i} dx^i$