Inner Product

SWBQ: Tell me something you know about the dot product

SWBQ: Tell me an algebraic property of the dot product.

$\{dx, dy, dz\} \longleftrightarrow \{\hat{x}, \hat{y}, \hat{z}\}$

$\hat{x}\cdot\hat{x} = 1 = \cdots$

$\hat{x}\cdot\hat{y} = 0 = \cdots$

$\therefore$ define $g(dx, dx) = 1 = \cdots$

$g(dx\ dy) = 0 = \cdots$

Def: An inner product g on a "vector space" V is a map $g : V \times V \longrightarrow \mathbb{R}$ such that

linear: $g(f\alpha + \beta, \gamma) = f\,g(\alpha, \gamma) + g(\beta, \gamma)$

symmetric: $g(\beta, \alpha) = g(\alpha, \beta)$

non-degenerate: $g(\alpha, \beta) = 0\ \forall \beta \Rightarrow \alpha = 0$

does not imply $g(\alpha, \alpha) \geq 0$

Polar Coordinates

$$r^2 = x^2 + y^2 \Longrightarrow r\,dr = x\,dx + y\,dy$$

$$\Longrightarrow g(r\,dr, r\,dr) = x^2 + y^2 = r^2$$

$$= r^2 g(dr, dr)$$

$$\tan^2\phi = \frac{y}{x} \Longrightarrow (1 + \tan^2\phi)\,d\phi = \frac{x\,dy - y\,dx}{x^2}$$

$$1 + \tan^2\phi = 1 + \frac{y^2}{x^2} = \frac{r^2}{x^2}$$

$$\Longrightarrow r^2 d\phi = x\,dy - y\,dx$$

$$\Longrightarrow g(r^2 d\phi, r^2 d\phi) = x^2 + y^2 = r^2$$

$$= r^4 g(d\phi, d\phi)$$

$$\therefore \quad g(dr, dr) = 1 = g(r\,d\phi, r\,d\phi)$$

$$g(dr, r\,d\phi) = g\left(\frac{x\,dx + y\,dy}{r}, \frac{x\,dy - y\,dx}{r}\right) = 0$$

$$\therefore \quad \boxed{\underset{\text{orthonormal}}{\{dx, dy\}} \Longleftrightarrow \underset{\text{orthonormal}}{\{dr, r\,d\phi\}}}$$