Curvature

Recall : $\boxed{\begin{array}{l} d\hat{e}_j = \omega^i{}_j\, \hat{e}_i \\ \hat{e}_i \cdot d\hat{e}_j = \omega_{ij} \end{array}}$

(I) $d\vec{r} = \sigma^j \hat{e}_j$

$$\Rightarrow d(d\vec{r}) = d\sigma^j \hat{e}_j - \sigma^j \wedge d\hat{e}_j$$
$$= d\sigma^i \hat{e}_i - \sigma^j \wedge \omega^i{}_j \hat{e}_i$$
$$= (d\sigma^i + \omega^i{}_j \wedge \sigma^j)\, \hat{e}_i$$

1st structure eq : $\boxed{\Theta^i = d\sigma^i + \omega^i{}_j \wedge \sigma^j} = 0$ (torsion-free)

torsion 2-forms

(II) $d\hat{e}_j = \omega^k{}_j \hat{e}_k$

$$\Rightarrow d(d\hat{e}_j) = d\omega^k{}_j\, \hat{e}_k - \omega^k{}_j \wedge d\hat{e}_k$$
$$= d\omega^i{}_j\, \hat{e}_i - \omega^k{}_j \wedge \omega^i{}_k\, \hat{e}_i$$
$$= (d\omega^i{}_j - \omega^k{}_j \wedge \omega^i{}_k)\, \hat{e}_i$$

2nd structure eq : $\boxed{\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j}$

curvature 2-forms

$$\therefore \boxed{\begin{array}{l} d^2\vec{r} = \Theta^i \hat{e}_i \\ d^2\hat{e}_j = \Omega^i{}_j\, \hat{e}_i \end{array}}$$

Notation : $\Omega^i{}_j = \frac{1}{2} R^i{}_{jkl}\, \sigma^k \wedge \sigma^l$

2 dimensions

Recall: $S = -\left(\Gamma^3{}_{ij}\right)$

plane: $S = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ cyl: $S = \begin{pmatrix} 1/r & 0 \\ 0 & 0 \end{pmatrix}$ sph: $S = \begin{pmatrix} 1/r & 0 \\ 0 & 1/r \end{pmatrix}$

extrinsic

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$$

$$\Rightarrow \Omega^1{}_2 = d\omega^1{}_2 + \cancel{\omega^1{}_1} \wedge \omega^1{}_2 + \omega^1{}_2 \wedge \cancel{\omega^2{}_2}$$

$$= d\omega^1{}_2$$

$$= K\, \sigma^1 \wedge \sigma^2$$

plane: $\omega^1{}_2 = 0 \Rightarrow K = 0$

cyl: $\omega^1{}_2 = -d\phi \Rightarrow d\omega^1{}_2 = 0$

$$\Rightarrow K = 0$$

sph: $\omega^1{}_2 = -\cos\theta\, d\phi$

$$\Rightarrow d\omega^1{}_2 = \sin\theta\, d\theta \wedge d\phi$$

$$= \frac{1}{r^2}\, \sigma^1 \wedge \sigma^2$$

$$\Rightarrow K = \frac{1}{r^2}$$

intrinsic

Theorema Egregium

"Outrageous Theorem"

$$\boxed{K = \det S}$$ in Euclidean $\mathbb{R}^3$

Pf: 3-d $\quad d\omega^1{}_2 + \omega^1{}_3 \wedge \omega^3{}_2 = 0$

$$\Rightarrow d\omega^1{}_2 = -\omega^1{}_3 \wedge \omega^3{}_2$$

$$= +\omega^3{}_1 \wedge \omega^3{}_2$$

$$= \Gamma^3{}_{1i}\,\sigma^i \wedge \Gamma^3{}_{2j}\,\sigma^j$$

$$= \left(\Gamma^3{}_{11}\Gamma^3{}_{22} - \Gamma^3{}_{12}\Gamma^3{}_{21}\right)\sigma^1 \wedge \sigma^2 + \text{terms involving } \sigma^3$$

$$= \det\left(\Gamma^3{}_{ij}\right)\ \sigma^1 \wedge \sigma^2$$

Bianchi Identities

$$0 = d^2\sigma^i = d(d\sigma^i) = d(-\omega^i{}_k \wedge \sigma^k)$$
$$= -d\omega^i{}_k \wedge \sigma^k + \omega^i{}_k \wedge d\sigma^k$$
$$= -d\omega^i{}_j \wedge \sigma^j + \omega^i{}_k \wedge (-\omega^k{}_j \wedge \sigma^j)$$
$$= -(d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j) \wedge \sigma^j$$
$$= -\Omega^i{}_j \wedge \sigma^j$$
$$\therefore \boxed{\Omega^i{}_j \wedge \sigma^j = 0}$$

1st Bianchi id

$$0 = d^2\omega^i{}_j = d(d\omega^i{}_j)$$
$$= d(\Omega^i{}_j - \omega^i{}_k \wedge \omega^k{}_j)$$
$$= d\Omega^i{}_j - d\omega^i{}_k \wedge \omega^k{}_j + \omega^i{}_k \wedge d\omega^k{}_j$$
$$= d\Omega^i{}_j - (\Omega^i{}_k - \cancel{\omega^i{}_m \wedge \omega^m{}_k}) \wedge \omega^k{}_j$$
$$+ \omega^i{}_k \wedge (\Omega^k{}_j - \cancel{\omega^k{}_m \wedge \omega^m{}_j})$$
$$\therefore \boxed{d\Omega^i{}_j + \omega^i{}_k \wedge \Omega^k{}_j - \Omega^i{}_k \wedge \omega^k{}_j = 0}$$

2nd Bianchi id