Example: Spherical Coordinates

$$\vec{r} = r\hat{r} \Rightarrow d\vec{r} = dr\,\hat{r} + r\,d\hat{r}$$

But $d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$

$$\Rightarrow \boxed{d\hat{r} = d\theta\,\hat{\theta} + \sin\theta\,d\phi\,\hat{\phi}}$$

Furthermore,

$$d\hat{\theta}\cdot\hat{r} = -d\hat{r}\cdot\hat{\theta} = -d\theta$$
$$d\hat{\phi}\cdot\hat{r} = -d\hat{r}\cdot\hat{\phi} = -\sin\theta\,d\phi$$
$$d\hat{\phi}\cdot\hat{\theta} = -d\hat{\theta}\cdot\hat{\phi} =: -\alpha$$

$$\therefore \boxed{\begin{aligned} d\hat{\theta} &= -d\theta\,\hat{r} + \alpha\,\hat{\phi} \\ d\hat{\phi} &= -\sin\theta\,d\phi\,\hat{r} - \alpha\,\hat{\theta} \end{aligned}}$$

To find α, need $d^2\vec{r} = \vec{0}$:

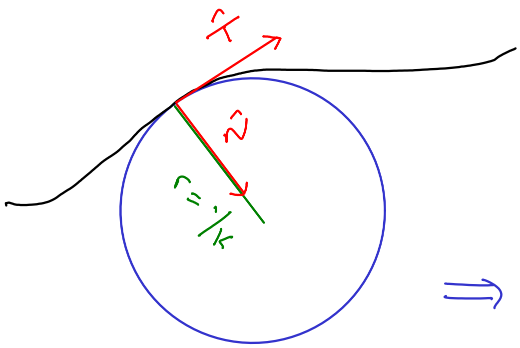
$$\begin{aligned} d(d\vec{r}) = &-dr\wedge d\hat{r} + dr\wedge d\theta\,\hat{\theta} - r\,d\theta\wedge d\hat{\theta} \\ &+ \sin\theta\,dr\wedge d\phi\,\hat{\phi} + r\cos\theta\,d\theta\wedge d\phi\,\hat{\phi} \\ &- r\sin\theta\,d\phi\wedge d\hat{\phi} \end{aligned}$$

$$\begin{aligned} = &-\cancel{dr\wedge d\theta\,\hat{\theta}} + \cancel{dr\wedge d\theta\,\hat{\theta}} - \cancel{\sin\theta\,dr\wedge d\phi\,\hat{\phi}} \\ &- r\,d\theta\wedge\alpha\,\hat{\phi} + \cancel{\sin\theta\,dr\wedge d\phi\,\hat{\phi}} \\ &+ r\cos\theta\,d\theta\wedge d\phi\,\hat{\phi} + r\sin\theta\,d\phi\wedge\alpha\,\hat{\theta} \end{aligned}$$

$$= 0 \quad \Rightarrow \quad \begin{aligned} d\phi\wedge\alpha &= 0 \\ d\theta\wedge\alpha &= \cos\theta\,d\theta\wedge d\phi \end{aligned}$$

$$\Rightarrow \alpha = \cos\theta\,d\phi$$

Curvature of a Curve



$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{ds}{dt}\hat{T} = v\hat{T}$$

$$\Rightarrow \quad d\vec{r} = ds\,\hat{T} = v\,dt\,\hat{T}$$

But $\hat{T}\cdot\hat{T} = 1$

$$\Rightarrow \quad d\hat{T}\cdot\hat{T} = 0$$

principal unit normal

$$\Rightarrow \quad d\hat{T} = \kappa\, ds\, \hat{N} = \kappa v\, dt\, \hat{N}$$

$$\therefore \quad d\vec{v} = dv\,\hat{T} + v\,d\hat{T}$$
$$= dv\,\hat{T} + \kappa v\, ds\,\hat{N}$$
$$= dv\,\hat{T} + \kappa v^2 dt\,\hat{N}$$

linear acceleration is $\frac{dv}{dt}$

centripetal acceleration is $\kappa v^2 = \frac{v^2}{r}$

κ = curvature

$$\boxed{\begin{aligned} \therefore \quad \kappa\, ds &= d\hat{T}\cdot\hat{N} \\ &= -d\hat{N}\cdot\hat{T} \end{aligned}}$$

Curvature of Surfaces

SWBQ: Give me a surface

A surface in $\mathbb{R}^3$ is $\{f = \text{const}\}$

use as x^3

$\therefore$ choose orthonormal basis $\{\hat{e}_1, \hat{e}_2, \hat{n}\}$, $\hat{n} = \hat{e}_3$

$\therefore$ curvature in $\hat{e}_i$ direction is

$$\kappa\, ds = -d\hat{n} \cdot \hat{e}_i$$
$$= -d\hat{e}_3 \cdot \hat{e}_i$$
$$= +\omega_{3i}$$

On surface $\Rightarrow x^3 = \text{const}$

$$\Rightarrow dx^3 = 0 \Rightarrow \sigma^3 = 0$$

$$\therefore \cancel{d\sigma^3} + \omega^3{}_1 \wedge \sigma^1 + \omega^3{}_2 \wedge \sigma^2 = 0$$

But $\omega^3{}_i = \Gamma^3{}_{i\partial}\, \sigma^\partial = \Gamma^3{}_{i1}\, \sigma^1 + \Gamma^3{}_{i2}\, \sigma^2 + \Gamma^3{}_{i3}\, \cancel{\sigma^3}$

$$\Rightarrow \Gamma^3{}_{12}\, \sigma^2 \wedge \sigma^1 + \Gamma^3{}_{21}\, \sigma^1 \wedge \sigma^2 = 0$$

$$\Rightarrow \Gamma^3{}_{12} = \Gamma^3{}_{21}$$

$\therefore$ $S = \left(-\Gamma^3{}_{i\partial}\right)$ is symmetric

shape operator

eigenvalues = principal curvatures (max & min)

trace = mean curvature

determinant = Gauss curvature