Levi-Civita Connection

Any $\omega^i{}_j$ leads to a connection with desired properties

∴ impose additional constraints

Require: ① $\boxed{d(\vec{v}\cdot\vec{w}) = d\vec{v}\cdot\vec{w} + \vec{v}\cdot d\vec{w}}$

metric compatible

② $\boxed{d^2\vec{r} = d(d\vec{r}) = 0}$

torsion free

NOT TRUE IN GENERAL!

① $\Rightarrow$

$$d(\hat{e}_i\cdot\hat{e}_j) = d\hat{e}_i\cdot\hat{e}_j + \hat{e}_i\cdot d\hat{e}_j$$

$$d(\hat{e}_i\cdot\hat{e}_j) = d(\delta_{ij}) = 0, \qquad d\hat{e}_i\cdot\hat{e}_j + \hat{e}_i\cdot d\hat{e}_j = \omega_{ji} + \omega_{ij}$$

② $\Rightarrow$

$$\begin{aligned} 0 &= d(\sigma^j\hat{e}_j) \\ &= d\sigma^j\,\hat{e}_j - \sigma^j\wedge d\hat{e}_j \\ &= d\sigma^i\,\hat{e}_i - \sigma^j\wedge\omega^i{}_j\,\hat{e}_i \\ &= (d\sigma^i + \omega^i{}_j\wedge\sigma^j)\,\hat{e}_i \end{aligned}$$

∴

$$\boxed{\begin{aligned} \omega_{ij} + \omega_{ji} &= 0 \\ d\sigma^i + \omega^i{}_j\wedge\sigma^j &= 0 \end{aligned}}$$

Levi-Civita connection

Thm: $\exists!$ connection satisfying ①&②

Idea: $\frac{n(n+1)}{2}n + n\frac{n(n-1)}{2} = n^3$ linear eqs in $n^2 n = n^3$ unknowns

Example: Rectangular Coordinates

$$\sigma^x = dx$$
$$\sigma^y = dy$$

$$\Rightarrow \quad d\sigma^x + \cancel{\omega^x{}_x} \wedge \sigma^x + \omega^x{}_y \wedge \sigma^y = 0$$
$$d\sigma^y + \omega^y{}_x \wedge \sigma^x + \cancel{\omega^y{}_y} \wedge \sigma^y = 0$$

$$\Rightarrow \quad \cancel{d(dx)} + \omega^x{}_y \wedge \sigma^y = 0$$
$$\cancel{d(dy)} + \omega^y{}_x \wedge \sigma^x = 0$$

$$\Rightarrow \quad \omega_{xy} \wedge \sigma^y = 0 = \omega_{xy} \wedge \sigma^x$$

$$\therefore \quad \boxed{\omega_{xy} = 0}$$

Example: Polar Coordinates

$$\sigma^r = dr$$
$$\sigma^\phi = r\,d\phi$$

$$\therefore \quad \underbrace{d\sigma^r}_{=0} + \cancel{\omega^r{}_r} \wedge \sigma^r + \omega^r{}_\phi \wedge \sigma^\phi = 0$$

$$\underbrace{d\sigma^\phi}_{= dr\wedge d\phi} + \omega^\phi{}_r \wedge \sigma^r + \cancel{\omega^\phi{}_\phi} \wedge \sigma^\phi = 0$$

$$\therefore \quad 0 + \omega^r{}_\phi \wedge r\,d\phi = 0$$
$$dr\wedge d\phi + \omega^\phi{}_r \wedge dr = 0$$

$$\therefore \quad \omega^r{}_\phi = 0\,dr + (\quad)\,d\phi$$
$$\omega^\phi{}_r = (\quad)\,dr + d\phi$$

But $\omega^r{}_\phi = \omega_{r\phi} = -\omega_{\phi r} = -\omega^\phi{}_r$

$$\Rightarrow \boxed{\omega_{r\phi} = -d\phi}$$

Enough to guess answer:

$$\omega^\phi{}_r \wedge dr = -dr\wedge d\phi = d\phi\wedge dr$$

so try $\omega^\phi{}_r = d\phi$

$$\Rightarrow \omega^r{}_\phi = -d\phi$$

Does this satisfy other eq? Yes!

$\therefore$ done!