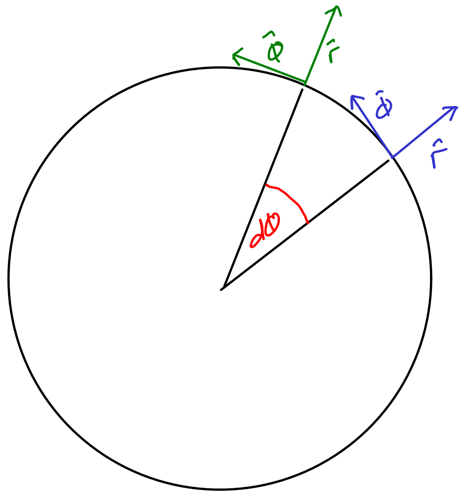
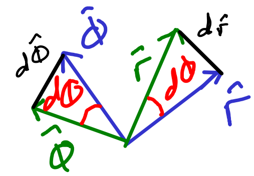
Polar Coordinates

SWBQ: Find $d\hat{r}, d\hat{\phi}$

I





$$\Rightarrow \boxed{\begin{aligned} d\hat{r} &= d\phi\,\hat{\phi} \\ d\hat{\phi} &= -d\phi\,\hat{r} \end{aligned}}$$

II

$$\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned} \Rightarrow \begin{aligned} \vec{r} &= x\hat{x} + y\hat{y} \\ &= r\cos\phi\,\hat{x} + r\sin\phi\,\hat{y} \end{aligned}$$

$$\Rightarrow d\vec{r} = \frac{\partial \vec{r}}{\partial r} dr + \frac{\partial \vec{r}}{\partial \phi} d\phi$$

$$\Rightarrow \hat{r} = \frac{\partial \vec{r}/\partial r}{|\partial \vec{r}/\partial r|} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\hat{\phi} = \frac{\partial \vec{r}/\partial \phi}{|\partial \vec{r}/\partial \phi|} = \frac{-r\sin\phi\,\hat{x} + r\cos\phi\,\hat{y}}{r}$$

$$\begin{aligned} \Rightarrow d\hat{r} &= -\sin\phi\, d\phi\,\hat{x} + \cos\phi\, d\phi\,\hat{y} \\ &= \hat{\phi}\, d\phi \\ d\hat{\phi} &= -\cos\phi\, d\phi\,\hat{x} - \sin\phi\, d\phi\,\hat{y} \\ &= -\hat{r}\, d\phi \end{aligned}$$

III

$$\vec{r} = r\hat{r} = x\hat{x} + y\hat{y}$$

$$\Rightarrow \hat{r} = \frac{x\hat{x} + y\hat{y}}{r} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\therefore r\hat{\phi} \perp r\hat{r} \Rightarrow r\hat{\phi} = \pm(-y\hat{x} + x\hat{y})$$

$$= -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

SWBQ:

Draw a curve.
Pick a point.
Draw $\vec{v}$ and $\dot{\vec{v}}$ at your point
Is $\dot{\vec{v}} \perp \vec{v}$?
Draw $\hat{T}$ and $\dot{\hat{T}}$ at your point
Is $\dot{\hat{T}} \perp \hat{T}$?

Idea: $|\hat{u}| = 1$

$$\Rightarrow \hat{u} \cdot \hat{u} = 1$$

$$\Rightarrow d(\hat{u} \cdot \hat{u}) = 0$$

$$d\hat{u} \cdot \hat{u} + \hat{u} \cdot d\hat{u} = 2 d\hat{u} \cdot \hat{u}$$

$\therefore d\hat{u} \perp \hat{u}$

IV $\quad \vec{r} = r\hat{r}$ in polar coords

$$\Rightarrow d\vec{r} = dr\,\hat{r} + r\,d\hat{r}$$

But $\quad d\vec{r} = dr\,\hat{r} + r\,d\phi\,\hat{\phi}$

$$\Rightarrow \underline{d\hat{r} = d\phi\,\hat{\phi}}$$

what about $d\hat{\phi}$?

$$d\hat{\phi} = (\quad)\hat{r} + (\quad)\hat{\phi}$$

$d\hat{\phi}\cdot\hat{r} \qquad d\hat{\phi}\cdot\hat{\phi}$

But $\quad |\hat{\phi}| = 1 \Rightarrow d\hat{\phi}\cdot\hat{\phi} = 0$

Furthermore,

$$\hat{\phi}\cdot\hat{r} = 0$$

$$\Rightarrow d(\hat{\phi}\cdot\hat{r}) = 0$$

$$\qquad \| $$

$$d\hat{\phi}\cdot\hat{r} + \hat{\phi}\cdot d\hat{r}$$

$$\therefore\ d\hat{\phi}\cdot\hat{r} = -d\hat{r}\cdot\hat{\phi} = -d\phi$$

$$\therefore\ \underline{d\hat{\phi} = -d\phi\,\hat{r}}$$

d on vectors

$\vec{F} = F^x \hat{x} + F^y \hat{y}$

$\Longrightarrow d\vec{F} = dF^x \hat{x} + dF^y \hat{y}$

because $d\hat{x} = 0 = d\hat{y}$

In general: $\vec{F} = F^i \hat{e}_i$

$\Longrightarrow d\vec{F} = dF^i + F^i d\hat{e}_i$

Properties

$(\alpha \in \Lambda^p,\ \vec{v}, \vec{w}\ \text{vf},\ a = \text{const})$

$$d(a\vec{v} + \vec{w}) = a\,d\vec{v} + d\vec{w}$$
$$d(\alpha\vec{v}) = d\alpha\,\vec{v} + (-1)^p \alpha \wedge d\vec{v}$$

$$\left[\quad d(\vec{v}\alpha) = d\vec{v} \wedge \alpha + \vec{v}\,d\alpha \right]$$

$\therefore$ Need $d\hat{e}_i$

Connection

Name what we don't know!

$$\boxed{d\hat{e}_j = \omega^i{}_j\, \hat{e}_i}$$

connection 1-forms

<u>Notation</u>

$$g_{ij} = \hat{e}_i \cdot \hat{e}_j$$ "metric"

$$(g_{ij})^{-1} = g^{ij}$$
$$= g(\sigma^i, \sigma^j)$$

$= I$ in orthonormal basis ($\&\ s=0$)

$$d\vec{r} = \sigma^i \hat{e}_i$$

$$ds^2 = d\vec{r} \cdot d\vec{r}$$
$$= g_{ij}\, \sigma^i \sigma^j$$

[similar notation for arbitrary basis]

$ds^2 = g_{ij}\, dx^i dx^j$ in coordinate basis

$$\omega_{ij} = \hat{e}_i \cdot d\hat{e}_j$$
$$= g_{ik}\, \omega^k{}_j$$

<u>Example</u>: Polar coordinates

$(\hat{e}_r = \hat{r},\ \hat{e}_\phi = \hat{\phi})$

$$d\hat{r} = d\phi\, \hat{\phi} = \omega^r{}_r\, \hat{r} + \omega^\phi{}_r\, \hat{\phi}$$
$$d\hat{\phi} = -d\phi\, \hat{r} = \omega^r{}_\phi\, \hat{r} + \omega^\phi{}_\phi\, \hat{\phi}$$

$$\Rightarrow \boxed{\begin{array}{c} \omega^r{}_r = 0 = \omega^\phi{}_\phi \\ \omega^\phi{}_r = d\phi = -\omega^r{}_\phi \end{array}}$$