MTH 434/534 **HW #6** Winter 2021

1. **SPHERICAL COORDINATES, II**

Consider spherical coordinates (r,θ,ϕ) and the adapted orthonormal basis

$$\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\} = \{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$$

The "infinitesimal displacement vector" $d\vec{\boldsymbol{r}}$ relates this basis to an orthonormal basis of 1-forms via

$$d\vec{\boldsymbol{r}} = dr\,\hat{\boldsymbol{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$$

Both sides of this equation are really ***vector valued 1-forms***.

WARNING: These conventions imply $\tan\phi = \frac{y}{x}$.

(a) Determine the exterior derivative of each basis vector (not 1-form) above, that is, compute $d\hat{\boldsymbol{r}}$, $d\hat{\boldsymbol{\theta}}$, and $d\hat{\boldsymbol{\phi}}$. *What sort of a beast should you get?*
HINT: What is the position vector $\vec{\boldsymbol{r}}$ *in this basis?*

You may calculate in any coordinate system, but the final answer should be entirely in terms of spherical coordinates and basis vectors. If you are having trouble getting started, see me.

(b) Compute $\omega_{ij} = \hat{\boldsymbol{e}}_i \cdot d\hat{\boldsymbol{e}}_j$ for $i,j = 1,2,3$. *What sort of a beast should you get?*

(c) Compute $\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$ for $i,j = 1,2,3$ (and where there is an implicit sum over k). *What sort of a beast should you get?*

NOTATION: One normally defines the connection 1-forms $\omega^i{}_j$ *implicitly through the equation* $d\hat{\boldsymbol{e}}_j = \omega^i{}_j\,\hat{\boldsymbol{e}}_i$. *In Euclidean signature (but NOT in general), the 1-forms* $\omega^i{}_j$ *and* ω_{ij} *are identical. For the purposes of part (c), you may therefore assume that* $\omega^i{}_j = \omega_{ij}$.