SPHERICAL COORDINATES

Tevian Dray

1 Coordinates

Consider spherical coordinates in 3-dimensional Euclidean space, given by [1]

$$
\begin{aligned}
x &= r \sin\theta \cos\phi, \\
y &= r \sin\theta \sin\phi, \\
z &= r \cos\theta.
\end{aligned}
$$

Zapping these expressions with d yields

$$
\begin{aligned}
dx &= \sin\theta\cos\phi\, dr + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi, \\
dy &= \sin\theta\sin\phi\, dr + r\cos\theta\sin\phi\, d\theta + r\sin\theta\cos\phi\, d\phi, \\
dz &= \cos\theta\, dr - r\sin\theta\, d\theta,
\end{aligned}
$$

from which it is straightforward to compute

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

so that the orthonormal basis of 1-forms in spherical coordinates is $\{dr, r\,d\theta, r\sin\theta\,d\phi\}$.

Alternatively, substituting the above expressions into

$$d\vec{\boldsymbol{r}} = dx\,\hat{\boldsymbol{x}} + dy\,\hat{\boldsymbol{y}} + dz\,\hat{\boldsymbol{z}}$$

and using

$$d\vec{\boldsymbol{r}} = \sum_i \frac{\partial\vec{\boldsymbol{r}}}{\partial u^i} du^i = \sum_i \frac{1}{h_i}\frac{\partial\vec{\boldsymbol{r}}}{\partial u^i}(h_i\,du^i)$$

with $u^i = r, \theta, \phi$ and

$$h_i = \left|\frac{\partial\vec{\boldsymbol{r}}}{\partial u^i}\right|$$

leads to the same result for the orthonormal basis of 1-forms, namely $h_r = 1$, $h_\theta = r$, and $h_\phi = r\sin\theta$, while also providing expressions for the orthonormal basis $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ in terms of $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$.

[1] WARNING: These are "physics" conventions: θ is the angle from the north pole (colatitude), and ϕ is the angle in the xy-plane (longitude).

Since we are in an orthonormal basis, it is straightforward to compute

$$\begin{aligned} *dr &= r\,d\theta \wedge r\sin\theta\,d\phi, \\ *r\,d\theta &= r\sin\theta\,d\phi \wedge dr, \\ *r\,d\phi &= dr \wedge r\,d\theta. \end{aligned}$$

2 Gradient

The gradient of any function f in spherical coordinates is given by

$$\begin{aligned} \nabla f := df &= \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi}d\phi \\ &= \frac{\partial f}{\partial r}dr + \frac{1}{r}\frac{\partial f}{\partial \theta}(r\,d\theta) + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}(r\sin\theta\,d\phi). \end{aligned}$$

3 Curl

Any 1-form can be expressed in an orthonormal, spherical basis as

$$\alpha = \alpha_r\,dr + \alpha_\theta\,r\,d\theta + \alpha_\phi\,r\sin\theta\,d\phi.$$

The curl of α is given by

$$\begin{aligned} \nabla \times \alpha := *d\alpha &= *d\big(\alpha_r\,dr + \alpha_\theta\,r\,d\theta + \alpha_\phi\,r\sin\theta\,d\phi\big) \\ &= *\big(d\alpha_r \wedge dr + d(\alpha_\theta r) \wedge d\theta + d(\alpha_\phi r\sin\theta) \wedge d\phi\big) \\ &= *\big(d\alpha_r \wedge dr + \frac{1}{r}\,d(\alpha_\theta r) \wedge r\,d\theta + \frac{1}{r\sin\theta}\,d(\alpha_\phi r\sin\theta) \wedge r\sin\theta\,d\phi\big) \\ &= *\Bigg(\left(\frac{1}{r}\frac{\partial \alpha_r}{\partial \theta}(r\,d\theta) + \frac{1}{r\sin\theta}\frac{\partial \alpha_r}{\partial \phi}(r\sin\theta\,d\phi)\right) \wedge dr \\ &\qquad + \frac{1}{r}\left(\frac{\partial (r\,\alpha_\theta)}{\partial r}\,dr + \frac{1}{r\sin\theta}\frac{\partial (r\,\alpha_\theta)}{\partial \phi}(r\sin\theta\,d\phi)\right) \wedge r\,d\theta \\ &\qquad + \frac{1}{r\sin\theta}\left(\frac{\partial (r\sin\theta\,\alpha_\phi)}{\partial r}\,dr + \frac{1}{r}\frac{\partial (r\sin\theta\,\alpha_\phi)}{\partial \theta}\,r\,d\theta\right) \wedge r\sin\theta\,d\phi\Bigg) \\ &= \left(-\frac{1}{r}\frac{\partial \alpha_r}{\partial \theta}\,r\sin\theta\,d\phi + \frac{1}{r\sin\theta}\frac{\partial \alpha_r}{\partial \phi}\,r\,d\theta\right) + \frac{1}{r}\left(\frac{\partial (r\,\alpha_\theta)}{\partial r}\,r\sin\theta\,d\phi - \frac{1}{r\sin\theta}\frac{\partial (r\,\alpha_\theta)}{\partial \phi}\,dr\right) \\ &\qquad + \frac{1}{r\sin\theta}\left(-\frac{\partial (r\sin\theta\,\alpha_\phi)}{\partial r}\,r\,d\theta + \frac{1}{r}\frac{\partial (r\sin\theta\,\alpha_\phi)}{\partial \theta}\,dr\right) \\ &= \frac{1}{r\sin\theta}\left(\frac{\partial (\sin\theta\,\alpha_\phi)}{\partial \theta} - \frac{\partial \alpha_\theta}{\partial \phi}\right)dr + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial \alpha_r}{\partial \phi} - \frac{\partial (r\,\alpha_\phi)}{\partial r}\right)r\,d\theta \\ &\qquad + \frac{1}{r}\left(\frac{\partial (r\,\alpha_\theta)}{\partial r} - \frac{\partial \alpha_r}{\partial \theta}\right)r\sin\theta\,d\phi \end{aligned}$$

4 Divergence

The divergence of α is given by

$$\begin{aligned}
\nabla \cdot \alpha := *d{*}\alpha &= *d{*}\big(\alpha_r\, dr + \alpha_\theta\, r\, d\theta + \alpha_\phi\, r \sin\theta\, d\phi\big) \\
&= *d\big(\alpha_r *dr + \alpha_\theta *r\, d\theta + \alpha_\phi *r \sin\theta\, d\phi\big) \\
&= *d\big(\alpha_r\, r\, d\theta \wedge r \sin\theta\, d\phi + \alpha_\theta\, r \sin\theta\, d\phi \wedge dr + \alpha_\phi\, dr \wedge r\, d\theta\big) \\
&= *\big(d(r^2 \sin\theta\, \alpha_r) \wedge d\theta \wedge\ d\phi + d(r \sin\theta\, \alpha_\theta) \wedge d\phi \wedge dr + d(r\, \alpha_\phi) \wedge dr \wedge d\theta\big) \\
&= *\left(\frac{\partial(r^2 \sin\theta\, \alpha_r)}{\partial r} + \frac{\partial(r \sin\theta\, \alpha_\theta)}{\partial \theta} + \frac{\partial(r\, \alpha_\phi)}{\partial \phi}\right) dr \wedge d\theta \wedge d\phi \\
&= \frac{1}{r^2 \sin\theta}\left(\frac{\partial(r^2 \sin\theta\, \alpha_r)}{\partial r} + \frac{\partial(r \sin\theta\, \alpha_\theta)}{\partial \theta} + \frac{\partial(r\, \alpha_\phi)}{\partial \phi}\right) \\
&= \frac{1}{r^2}\frac{\partial(r^2 \alpha_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial(\sin\theta\, \alpha_\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial(\alpha_\phi)}{\partial \phi}.
\end{aligned}$$