Differential Forms

① Consider $\mathbb{R}^n$ with coordinates $\{x^i\}$. (can be any surface in $\mathbb{R}^n$)

② Consider the formal objects $\{dx^i\}$.

Idea: $df = \frac{\partial f}{\partial x^i} dx^i = \frac{\partial f}{\partial u^i} du^i$

$\therefore$ treat $\{dx^i\}$ & $\{du^i\}$ as bases & use calculus to change basis!

③ Let V be the "almost vector space"

$$V = \langle dx^i \rangle = \{a_i\, dx^i\}$$

← Einstein summation convention!

"smooth" functions

What are the elements of V?

$$df = \frac{\partial f}{\partial x^i} dx^i = \vec{\nabla} f \cdot d\vec{r}$$

"integrand of line integral corresponding to $\vec{\nabla} f$"

Consistent: $d(x^a) = \frac{\partial x^a}{\partial x^i} dx^i$

$$= \delta^a_i\, dx^i = dx^a$$

Kronecker delta

The elements of V are 1-forms.

We write:

$$\boxed{V = \Lambda^1(\mathbb{R}^n) = \Lambda^1}$$

Products of Differential Forms

$\alpha, \beta \in \Lambda^1 \Rightarrow \alpha \wedge \beta \in \Lambda^2$

Properties: $\beta \wedge \alpha = -\alpha \wedge \beta$

$(\alpha, \beta, \gamma \in \Lambda^1)$ $\quad \alpha \wedge \alpha = 0$

$$(\alpha + \beta) \wedge \gamma = \alpha \wedge \gamma + \beta \wedge \gamma$$

$$f(\alpha \wedge \beta) = (f\alpha) \wedge \beta$$

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$$

Basis for 2-forms Λ^2:

$$\{dx^i \wedge dx^j\} \qquad 1 \le i < j \le n\,!$$

Basis for p-forms Λ^p: $\quad 0 \le p \le n$

$$\{dx^{i_1} \wedge \cdots \wedge dx^{i_p}\} \qquad 1 \le i_1 < \cdots < i_p \le n$$

How many (independent) p-forms?

$$\dim \Lambda^p = \binom{n}{p} = \frac{n!}{p!\,(n-p)!} = \binom{n}{n-p}$$

$$\Rightarrow \dim \Lambda^0 = 1 = \dim \Lambda^n$$

$$\& \quad \dim \Lambda^p = 0 \quad \text{for } p > n$$

Products of Forms, II

$\alpha \in \Lambda^p,\ \beta \in \Lambda^q \Rightarrow \alpha \wedge \beta \in \Lambda^{p+q}$

What is $\beta \wedge \alpha$?

Ex: $\alpha = dx$
$\beta = dy \wedge dz$

$\vdots$

$$\boxed{\beta \wedge \alpha = (-1)^{pq} \alpha \wedge \beta}$$

Why?