Cylindrical Coordinates

$$\boxed{\vec{dr} = dr\,\hat{r} + r\,d\Phi\,\hat{\Phi} + dz\,\hat{z}}$$

$\therefore$ orthonormal basis is $\{dr, r\,d\Phi, dz\}$

Parametrization

$$\boxed{\vec{r} = r\cos\Phi\,\hat{x} + r\sin\Phi\,\hat{y} + z\,\hat{z}}$$

$$\Rightarrow d\vec{r} = \underline{dr}\,(\underline{\cos\Phi\,\hat{x} + \sin\Phi\,\hat{y}})$$
$$+ \underline{r\,d\Phi}\,(\underline{-\sin\Phi\,\hat{x} + \cos\Phi\,\hat{y}})$$
$$+ \underline{dz}\;\underline{\hat{z}}$$

orthonormal basis of 1-forms

orthonormal basis of vector fields

$$\Rightarrow \hat{e}_r = \frac{\frac{\partial\vec{r}}{\partial r}}{\left|\frac{\partial\vec{r}}{\partial r}\right|} = \cos\Phi\,\hat{x} + \sin\Phi\,\hat{y} = \hat{r}$$

$$\hat{e}_\Phi = \frac{\frac{\partial\vec{r}}{\partial\Phi}}{\left|\frac{\partial\vec{r}}{\partial\Phi}\right|} = \frac{-\cancel{r}\sin\Phi\,\hat{x} + \cancel{r}\cos\Phi\,\hat{y}}{\cancel{r}} = \hat{\Phi}$$

$$\hat{e}_z = \frac{\frac{\partial\vec{r}}{\partial z}}{\left|\frac{\partial\vec{r}}{\partial z}\right|} = \hat{z}$$

Cylindrical Coordinates

$$\boxed{d\vec{r} = dr\,\hat{r} + r\,d\phi\,\hat{\phi} + dz\,\hat{z}}$$

$$\therefore\ df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \phi}d\phi + \frac{\partial f}{\partial z}dz$$

$$= \frac{\partial f}{\partial r}dr + \frac{1}{r}\frac{\partial f}{\partial \phi}r\,d\phi + \frac{\partial f}{\partial z}dz$$

$$= \vec{\nabla} f \cdot d\vec{r}$$

$$\Rightarrow \boxed{\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}}$$

$$\Rightarrow \vec{\nabla}\phi = \frac{1}{r}\hat{\phi}$$

$$\Rightarrow \vec{\nabla}\times\left(\frac{1}{r}\hat{\phi}\right) = \vec{\nabla}\times\vec{\nabla}\phi = \vec{0}$$

$$\Rightarrow \vec{\nabla}\times(f\hat{\phi}) = \vec{\nabla}\times\left((rf)\frac{\hat{\phi}}{r}\right)$$

$$= \vec{\nabla}(rf)\times\frac{\hat{\phi}}{r} + 0$$

$$= \left(\frac{\partial(rf)}{\partial r}\hat{r} + \cancel{\frac{1}{r}\frac{\partial(rf)}{\partial\phi}\hat{\phi}} + \frac{\partial(rf)}{\partial z}\hat{z}\right)\times\frac{\hat{\phi}}{r}$$

$$= \frac{1}{r}\frac{\partial(rf)}{\partial r}\hat{z} - \frac{\partial f}{\partial z}\hat{r}$$

Similarly, $\vec{\nabla}\times(\phi\hat{z}) = \vec{\nabla}\phi\times\hat{z} + \cancel{\phi\,\vec{\nabla}\times\hat{z}}$

$$= \frac{1}{r}\hat{\phi}\times\hat{z} = \frac{1}{r}\hat{r}$$

$$\Rightarrow \vec{\nabla}\cdot\left(\frac{1}{r}\hat{r}\right) = \vec{\nabla}\cdot\vec{\nabla}\times(\phi\hat{z}) = 0$$

$$\Rightarrow \vec{\nabla}\cdot(f\hat{r}) = \vec{\nabla}\cdot\left((rf)\frac{\hat{r}}{r}\right)$$

$$= \vec{\nabla}(rf)\cdot\frac{\hat{r}}{r} + 0$$

$$= \frac{1}{r}\frac{\partial(rf)}{\partial r}$$

Orthogonal Coordinates ($\mathbb{R}^3$)

"coordinate directions mutually $\perp$"

coordinates (u,v,w) orthogonal

$$\Rightarrow \quad d\vec{r} = h_u\, du\, \hat{u} + h_v\, dv\, \hat{v} + h_w\, dw\, \hat{w}$$

$$ds^2 = d\vec{r} \cdot d\vec{r} = h_u^2\, du^2 + h_v^2\, dv^2 + h_w^2\, dw^2$$

$\therefore \{\hat{u}, \hat{v}, \hat{w}\}$ is orthonormal basis for vector fields

$\{h_u\, du, h_v\, dv, h_w\, dw\}$ is orthonormal basis for Λ^1

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u} du + \frac{\partial \vec{r}}{\partial v} dv + \frac{\partial \vec{r}}{\partial w} dw$$

$$\Rightarrow \quad h_u \hat{u} = \frac{\partial \vec{r}}{\partial u}$$

$$\Rightarrow \quad h_u = \left| \frac{\partial \vec{r}}{\partial u} \right| \qquad \left(\begin{array}{c} h > 0 \text{ even} \\ \text{if } s \neq 0 \end{array} \right)$$

$$\hat{u} = \frac{1}{h_u} \frac{\partial \vec{r}}{\partial u}$$