Bases for Differential Forms

Let M be an n-dimensional surface with coordinates $\{x^i\}$ (not always $\mathbb{R}^n$)

$\Rightarrow \{dx^i\}$ is a basis for $\Lambda^1 = \Lambda^1(M)$

Suppose $\{\sigma^i\}$ is a basis for Λ^1

then $\{\sigma^I\} = \{\sigma^{i_1} \wedge \cdots \wedge \sigma^{i_p}\}$ is a basis for Λ^p

with $1 \le i_1 < \cdots < i_p \le n$

metric: $g^{ij} = g(\sigma^i, \sigma^j)$

$g \longleftrightarrow (g^{ij})$

really g^{-1}

symmetry: $g^{ji} = g^{ij}$

non-degenerate: $|g^{ij}| \neq 0$

- symmetric matrices can be diagonalized
- nonzero determinant $\Rightarrow$ no zeros on diagonal

$\therefore$ can rescale basis so diagonal elements are ± 1

orthonormal basis

$$\boxed{g(\sigma^i, \sigma^j) = \pm \delta^{ij}}$$

Coordinate basis	Orthonormal basis
$\{dx, dy\}$	$\{dx, dy\}$
$\{dr, d\phi\}$	$\{dr, r\,d\phi\}$

$$d\vec{r} = dx\,\hat{x} + dy\,\hat{y}$$
$$= dr\,\hat{r} + r\,d\phi\,\hat{\phi}$$

$$ds^2 = d\vec{r}\cdot d\vec{r}$$
$$= dx^2 + dy^2$$
$$= dr^2 + r^2 d\phi^2$$

In general:

$$\boxed{d\vec{r} = \sigma^i \hat{e}_i}$$

"vector-valued 1-form"

$$ds^2 = g_{ij}\,dx^i dx^j \qquad ds^2 = g_{ij}\,\sigma^i \sigma^j$$

"rank 2 tensor"

symmetric! ($dx\,dy = dy\,dx$)

"line element"

metric tensor: $g \longleftrightarrow (g_{ij})$

$g = g^{-1}$ if orthonormal

Convention: give line element

$\longmapsto$ can read off orthonormal basis!

Signature

\# of + & − signs in dpt of basis

The signature s of the metric is the number of minus signs

$s = m$ (many authors use $s = p - m$)

Euclidean : $s = 0$

Lorentzian : $s = 1$

Ex: Euclidean 2-space $(\mathbb{R}^2, \mathbb{E}^2)$

$$ds^2 = dx^2 + dy^2$$

Ex: Minkowski 2-space $(\mathbb{M}^2)$

$$ds^2 = dx^2 - dt^2$$

	flat	Curved
$s=0$	Euclidean	Riemannian
$s=1$	Minkowskian	Lorentzian