Integration by Parts

Recall: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$

$$\Longrightarrow \int_R d\alpha \wedge \beta = \int_R d(\alpha \wedge \beta) - (-1)^p \int_R \alpha \wedge d\beta$$

$$= \int_{\partial R} \alpha \wedge \beta - (-1)^p \int_R \alpha \wedge d\beta$$

Special cases

$$\int_c f\,dg = \int_c d(gf) = \overset{\int_{\partial c} gf}{\overset{\parallel}{gf}} - \int_c g\,df$$

$$\int_R f\,d\alpha = \int_{\partial R} f\alpha - \int_R df \wedge \alpha$$

Application

$$\int_{S^2} h\, d{*}dh = \int_{S^2} d(h{*}dh) - \int_{S^2} dh \wedge {*}dh$$

$$= \int_{\cancel{\partial S^2}_{=0}} h{*}dh - \int_{S^2} g(dh, dh)\,\omega$$

suppose $\Delta h = 0$ on S^2

$\Rightarrow {*}d{*}dh = 0$ on S^2

$\Rightarrow d{*}dh = 0$ on S^2

$\Rightarrow LHS = 0$

$\Rightarrow RHS = 0$

$\Rightarrow \int_{S^2} \underbrace{g(dh, dh)}_{\geq 0}\,\omega = 0$

$\Rightarrow g(dh, dh) = 0$ on S^2

$\Rightarrow dh = 0$ on S^2

$\Rightarrow h = \text{const}$!

$$\boxed{\text{The only solutions of } \Delta h = 0 \text{ on } S^2 \text{ is } h = \text{const}}$$